Solitary surface acoustic waves

1. The main target and objectives. The main target of the research that has been carried out at LGCIE INSA de Lyon for several years is devoted to constructing a mathematical model for the soliton-like surface acoustic waves (SAW) traveling in layered anisotropic media. These newly observed waves play a very important role in different areas of physical nature, and technology: from soliton-like seismic waves to SAW in nanoscale materials.

The geotechnical group in LGCIE has vast and recognized experience in soil mechanics, and the proposed research on soliton-like waves is in the course of these studies, as in future it is foreseen to integrate the developed mathematical model for soliton-like waves with the porous-mechanical models studied in LGCIE.

2. Basic concepts. *Solitons*, or by the original terminology *waves of translation*, were for the first time observed and described in [1] as a special kind of hydrodynamic waves that can arise and propagate in narrow channels. These hydrodynamic solitons satisfy the following conditions:

(i) These are the solitary waves, resembling propagation of the wave front of a shock wave;

(ii) These waves can propagate without considerable attenuation;

(iii) They do not change their form;

(iv) They propagate without diminution of speed (see, [2]).

It was shown later on, that motion of these waves is described by a non-linear Korteveg – de Vries (KdV) differential equation [3 – 5].

3. Studies of solitary waves in solids. In the following review we analyze solitary surface acoustic waves (SAW) propagating in elastic plates and rods at vanishing frequency $\omega \to 0$, or in terms of the wave number r, at $r \to 0$. The vanishing frequency SAW satisfy conditions (i) – (iv), and thus, by their properties, these waves resemble solitons in hydrodynamics. But in contrast to the genuine solitons, solitary waves are described by a *linear* vectorial differential equation, known as the Christoffel equation for Lamb waves.

Studies of the vanishing frequency solitary waves in plates with the finite phase speed have quite a long history. Presumably, the first asymptotic analysis of these waves propagating in *isotropic* plates was performed in [6], where an analytical expression for the phase speed of the solitary Lamb wave was obtained. In the course of analytical and numerical studies the limiting low frequency Lamb waves were also analyzed in [7 – 16]. However, polarization of these waves was not obtained. It should also be noted that for anisotropic (monoclinic) plates neither analytical expressions for the limiting wave speed c_S at $\omega \to 0$, nor polarization of the limiting wave, were obtained.

In the framework of the *linear* differential equations derived by Pochhammer [17] and Chree [18], the lower mode Pochhammer – Chree longitudinal and torsional waves propagating in *isotropic* circular cylinders at low and vanishing frequency, and thus resembling the solitons, were analyzed in [19 – 22].

The vanishing frequency waves in a circular cylinder were also studied in [23 – 25], as the solutions of a *nonlinear* differential equation similar to the KdV equation.

Analytical and numerical data obtained in [6 – 16, 19 – 22] reveal that in the vicinity of the limiting phase speed c_S at which the *linear* solitary waves propagate, the corresponding dispersion dependence $c(\omega)$ satisfies a condition

$$\left| c(\omega) - c_S \right| = O(\omega^n), \quad \omega \to +0, \tag{1}$$

where $c(\omega)$ is the phase speed considered as a function of frequency; and $n > 0$ is some positive number. However, numerical analyses in [19 – 22] did not allow one to define the exponent n.

The recent *asymptotic* analysis undertaken in LGCIE INSA de Lyon, allowed us to find both exponent n and obtain the limiting speed c_S value for the SH waves propagating in a two-layered plate; see [35, 36]. Our future studies are targeted to developing an *asymptotic* expansion method similar to one developed for the SH wave analysis, for obtaining the limiting wave speed and polarization of the solitary Lamb waves propagating in the multilayered plates.

4. Importance of studies on propagation of solitary waves in solids. Low or vanishing frequencies of the surface acoustic waves (SAW) traveling with the phase speed satisfying condition (1), need in a very small amount of energy for their excitation. Indeed, the specific kinetic energy is determined by the following expression:

$$E_{kin} \equiv \frac{1}{2}\rho\left|\dot{\mathbf{u}}\right|^2 = \frac{1}{2}\rho\left|\mathbf{m}\right|^2\omega^2, \tag{2}$$

where $\mathbf{m}$ is the wave amplitude (possibly varying along depth of a layer or a halfspace). The right-hand side of (2) ensures that at the finite values of the amplitudes and at $\omega \to 0$, the specific kinetic energy vanishes. It can be shown that the specific potential energy is also proportional to the square of amplitude and frequency, thus, vanishing at $\omega \to 0$, too.

Importance of these waves is also underlined by the fact that they resemble propagation of the wave front (WF) in a layer (see [26, Ch.V, §1] for the WF in terms of Hörmander's definition and [27, Ch.IV, §4.5] for the propagating WF in terms of mechanical applications). Thus, the limiting speeds c_s correspond to propagation of the wave fronts formed by either SH or Lamb waves.

5. Methods of analysis. Following pioneering Lamb work [28], the displacement field of any SAW traveling in *elastic* layer can be represented by

$$\mathbf{u}(\mathbf{x},t) = \left(\sum_{p=1}^{6}\mathbf{m}_p C_p e^{ir\gamma_p x'}\right)e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)}, \tag{3}$$

where $\mathbf{u}$ is the displacement field, $\mathbf{m}_p \in \mathbb{R}^3$ are the unit amplitudes (polarizations). It is assumed that each vector $\mathbf{m}_p$ belongs to the sagittal plane. This plane is determined by the unit normal $\mathbf{w} = \mathbf{n} \times \mathbf{\nu}$, where $\mathbf{n}$ is the unit normal to the wave front and $\mathbf{\nu}$ is the unit normal to the median plane of the layer. In representation (3) $x' \equiv \mathbf{\nu}\cdot\mathbf{x}$ is a coordinate along vector $\mathbf{\nu}$; r is the wave number; c is the phase speed; t is time, and γ_p are the Christoffel parameters obtained from equations of motion. In representation (3)

$$\mathbf{u}^p(\mathbf{x},t) = \mathbf{m}_p e^{ir\gamma_p x'} e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \quad (4)$$

are partial waves. The unknown coefficients C_p in (3) are determined up to a multiplier by the traction-free boundary conditions:

$$x' = \pm h: \qquad \mathbf{t}_\nu \equiv \boldsymbol{\nu}\cdot\mathbf{C}\cdot\cdot\nabla_\mathbf{x}\mathbf{u} = 0, \quad (5)$$

where $\mathbf{C}$ is the fourth-order elasticity tensor (for isotropic medium tensor $\mathbf{C}$ is determined by two independent constants); and $2h$ the depth of the layer. Exponential multiplier $e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)}$ in (3) and (4) stands for propagation of the plane wave front $\mathbf{n}\cdot\mathbf{x} = const$.

Remark. Representation (3) is valid in a case of the arbitrary *anisotropic* layer. In a case of isotropic layer, or a layer with monoclinic symmetry, representation (3) can be simplified by choosing the terminate value 4, instead of 6 at summation in (3). For SH and Love waves traveling in isotropic layers or layers with monoclinic symmetry, the corresponding terminate value in (3) is 2.

If a multilayered plate is considered, the solution is generally constructed by one of the following two methods: (i) the transfer matrix (TM) method, known also as Thomson – Haskell method due to its originators [30, 31]; and (ii) the global matrix (GM) method [32, 33].

The TM method is based on a sequential solution of the boundary-value problems on the interfaces and constructing the transfer matrices. A modification of this method suitable for asymptotic analyzing solitary waves propagating in the multilayered plates was developed recently by members of the research group in the LGCIE INSA de Lyon; see [34, 35]. The GM method is based on solving a system of the second-order equations with the piecewise-constant coefficients, resulting in constructing the special "global matrix", while being well suited for numerical computations, the GM method appears to be less suitable for asymptotic studies.

And, concluding a review of the mathematical methods used for analyzing linear solitary waves in plates, we should note another method, known also as the "complex six-dimensional formalism". Actually, this term is referred to a group of several different techniques of reducing the second-order differential equations of wave dynamics to a six-dimensional system of equations of the first order. In theoretical physics such a technique is called as the generalized Hamiltonian formalism. The members of the LGCIE INSA de Lyon team worked out a variant of the complex six-dimensional formalism that is ideally suited for asymptotic analysis of the solitary waves; see [35, 36].

6**. Obtained theoretical results in 2005-2008 years.** The main theoretical results in the solitary acoustic wave propagation gained by the team are:

– Developing a variant of the complex six-dimensional formalism for analyses of the SAW propagating in both isotropic and anisotropic layers;
– Formulating and proving theorem on existence of Love waves propagating in anisotropic layer in a contact with anisotropic substrate;
– Developing an asymptotic method for studying propagation of the solitary SH and Love waves in layered anisotropic media with monoclinic symmetry;
– Obtaining analytical expressions for the limiting SH and Love wave speeds at which the corresponding solitary waves propagate;

7. Anticipated theoretical results in 2009-2011 years. The main theoretical results that are planning to be obtained in the course of the theoretical studies are:

– Performing preliminary analysis for proving a theorem of existence (or non-existence) for Rayleigh-Lamb waves propagating in anisotropic layer(s) in a contact with anisotropic substrate;
– On the basis of the theorem of existence developing engineering solutions for creating wave barriers against Rayleigh-Lamb waves;
– On the basis of previous studies, developing an asymptotic method for analyzing propagation of the solitary Lamb and Rayleigh-Lamb waves in layered anisotropic media with monoclinic symmetry;
– Obtaining analytical expressions for the limiting Lamb and Rayleigh-Lamb wave speeds at which the corresponding solitary waves propagate;

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