EMMME
INSA de Lyon

Lecture course

Mathematical methods for vibration analyses

by

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I. Trigonometric, exponential, and hyperbolic functions

1.1. Basic properties of trigonometric functions

Definitions 1.1.1. for sine and cosine functions:
(1) These are functions defined by the following series:

$$\sin(x) \equiv \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^5}{5!} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

$$\cos(x) \equiv 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n}}{(2n)!}$$

(2) These are the solutions of the following differential equation:

$$\left(\frac{d^2}{dx^2} + 1 \right) f(x) = 0$$

Theorem 1.1.1

Power series in the right-hand side of (1) converge everywhere in $(-\infty, \infty)$, and both sine and cosine are (real) analytic functions

Definition 1.1.2

A function is analytic (in a particular vicinity), if it can be expanded into a power series, convergent in that vicinity.

Remark 1.1.1

From definitions (1) and (2) it can be difficult to deduce that both sine and cosine functions are periodic ones.

Remark 1.1.2.

Taylor's series in the right-hand sides of expressions (1) demonstrate that the round off error at numerical computations increases with the increase of $|x|$.

1.2. Introduction of a complex variable

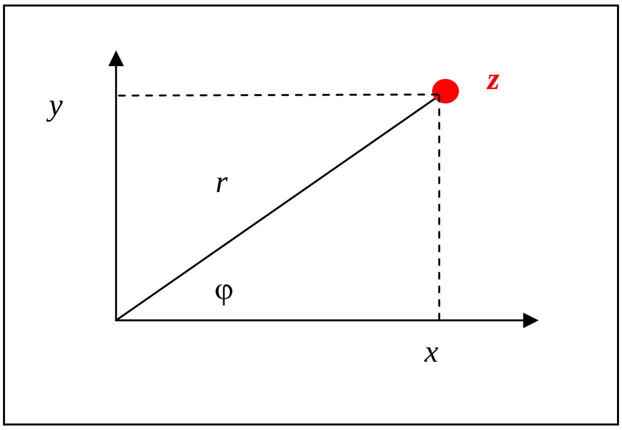
Definition 1.2.1.

A complex variable ***z*** is a variable that can be represented by

$$z = x + iy$$

where $i = \sqrt{-1}$; x is a real part, and y is an imaginary part,
denoted also by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$

Geometric representation for a complex variable.



$$z = r(\cos\varphi + i\sin\varphi) \qquad \text{where } \varphi \in [0;\, 2\pi)$$

Other useful formulas:

$$r \equiv |z| = \sqrt{x^2 + y^2}\ ; \qquad \varphi = \arctan(y/x)$$

A complex conjugate variable.

$$\bar{z} = x - iy$$

$$\bar{z} = r(\cos\varphi - i\sin\varphi)$$

Question 1.2.1.

In a geometrical representation, where does the complex conjugate variable lie?

Module of a complex variable.

$$|z| \equiv \sqrt{x^2 + y^2} = r = \sqrt{z\bar{z}}$$

Basic properties of complex variables

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 z_2 = (x_1 x_2) - (y_1 y_2) + i(x_1 y_2 + x_2 y_1) =$$
$$= r_1 r_2 \left(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right)$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{r_1}{r_2}\left(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right)$$

$$z^p = (x + iy)^p = r^p \left(\cos(p\varphi) + i\sin(p\varphi)\right)$$

Remark 1.2.1.

It should be noted that generally speaking raising to a power p can be ambiguous, as the following examples show

Examples 1.2.1.

(1) $\sqrt{i} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

and another value

$$\sqrt{i} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$$

(2) $\sqrt{z} = \sqrt{r}\left(\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\right)$

and another value

$$\sqrt{z} = \sqrt{r}\left(\cos\left(\frac{\varphi}{2} + \pi\right) + i\sin\left(\frac{\varphi}{2} + \pi\right)\right)$$

(3) This example gives *n* values for the *n*-th root

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right)\right), \qquad k = 0,\ldots n-1$$

1.3. Exponential function of a complex variable

Definition 1.3.1. for the exponential function

$$(1) \quad \exp(x) \equiv 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

or the exponent is a function satisfying the following equation:

$$(2) \quad \left(\frac{d^2}{dx^2} - 1 \right) f(x) = 0$$

Theorem 1.3.1.

a) Power series in the right-hand side (1) converges everywhere in $(-\infty; \infty)$

b) Exponential function is (real) analytic everywhere in $(-\infty; \infty)$

Remark 1.3.1.

In representation (1) an independent variable *x* can be complex.

Trigonometric representation for the exponential function:

(3) $\exp(z) = e^{x}(\cos y + i \sin y)$, where $z = x + iy$

since also

(4) $z = r(\cos \varphi + i \sin \varphi)$

we arrive at

(5) $\exp(z) = e^{r\cos\varphi}(\cos\left(r\sin\varphi\right) + i \sin\left(r\sin\varphi\right))$

1.4. Hyperbolic functions

Definition 1.4.1. for hyperbolic **sine** and **cosine** functions

$$(1)\qquad \begin{aligned} \sinh(x) &\equiv \frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots=\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!} \\ \cosh(x) &\equiv 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \end{aligned}$$

Both **sinh** and **cosh** satisfy the following differential equation:

$$(2)\qquad \left(\frac{d^2}{dx^2}-1\right)f(x)=0$$

Theorem 1.4.1.

a) Power series in the right-hand side (1) converge everywhere in $(-\infty;\,\infty)$

b) Both hyperbolic functions are (real) analytic in $(-\infty;\,\infty)$

Basic properties of hyperbolic functions

(3) $\sinh(x) = \dfrac{e^x - e^{-x}}{2}; \qquad \cosh(x) = \dfrac{e^x + e^{-x}}{2}$

(4) $\cosh^2(x) - \sinh^2(x) = 1$

(5) $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$

(6) $\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$

(7) $\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$

(8) $\dfrac{d}{dx}\sinh(x) = \cosh(x); \qquad \dfrac{d}{dx}\cosh(x) = \sinh(x)$

Remark 1.4.1.

In representation (1) an independent variable *x* can be complex.

Relations between hyperbolic, exponential, and trigonometric functions

(9) $$\sinh(z) = \frac{e^{z} - e^{-z}}{2}; \qquad \cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

(10) $$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}; \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

(11) $$\sin(z) = -i\sinh(iz); \qquad \cos(z) = \cosh(iz)$$

(12) $$\exp(iz) = \cos(z) + i\sin(z)$$

II. Matrix algebra

2.1. Basic definitions (matrix, eigenvalue, and eigenvector)

Definition 2.1.1. A matrix is a digital table of the form:

(1) $$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

In a definition above the matrix **A** is a rectangular one, the following matrix is a square one:

(2) $$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Definition 2.1.2.

Transposition of a matrix
(can be applied to an arbitrary matrix)

$$(3) \qquad \mathbf{B}^t = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}$$

or if the matrix is not a rectangular one

$$(4) \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

then

$$(5) \quad \mathbf{A}^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{pmatrix}$$

Multiplication of two matrices

(defined for two rectangular matrices $n \times m$ and $m \times p$)

(6) $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{32} \end{pmatrix}$$

Matrix C, called the product of matrices A and B will have dimension $n \times p$ (matrices –multipliers have dimension $n \times m$ and $m \times p$ respectively)

Definition 2.1.3.

The following formula defines components of the matrix-product in terms of the components of two matrices-multipliers:

(7) $$c_{rs} = \sum_{k=1}^{m} a_{rk} b_{ks}$$

If the two middle numbers don't match, you cannot multiply the matrices

Corollary.

Multiplication for **square** matrices is well defined, provided both matrices have the **same** dimension

Theorem. 2.1.1.

If multiplication of two matrices A and B is well-defined, then

(8) $\left(A\cdot B\right)^t = B^t \cdot A^t$

Exercise 2.1.1.

Perform multiplication

a) $\begin{pmatrix} 1 & 5 \\ 2 & 4 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 6 & 0 \\ 3 & 7 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 5 \\ 2 & 4 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 5 & 4 & 2 \\ 1 & 5 & 6 \\ 3 & 2 & 3 \end{pmatrix}$

Definition 2.1.4.

Integer power of a square matrix
(well-defined for any square matrix,
not defined for a rectangular matrix)

$$\mathbf{B}^n =$$

$$= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \underbrace{\cdot \cdots \cdot}_{n \ \ times} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{9}$$

Exercise 2.1.2.

Compute

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2$

(b) $\begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 0 & 1 \end{pmatrix}^3$

Definition 2.1.5.

The inverse matrix $\mathbf{A}^{-1}$
(well defined for **some** square matrices, **not defined for rectangular matrices**)

(10) $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$

where $\mathbf{I}$ is the unit matrix:

$$(11) \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Remark 2.1.1.

Even for a diagonal matrix the inverse matrix may not exist:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 2.1.6.

for the right eigenvector (right principle vector):

$$(12)\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

or

$$\mathbf{B} \cdot \overrightarrow{\mathbf{v}_r} = \lambda \overrightarrow{\mathbf{v}_r}$$

Definition 2.1.7.

for the left eigenvector (left principle vector):

$$(13) \qquad \overrightarrow{\mathbf{v}_l} \cdot \mathbf{B} = \lambda \overrightarrow{\mathbf{v}_l}$$

Remark 2.1.2.

Eigenvalues are defined only for **square** matrices

Definition 2.1.8.

An eigenvalue can be defined in two ways:

1) It is a multiplier λ in expressions (12), (13)

2) It is a root of the equation

$$(14) \qquad \det\left(\mathbf{B} - \lambda \mathbf{I}\right) = 0$$

Question. 2.1.1.

What type of equation does the left-hand side of Eq. (14) is?

Can the eigenvalue be zero?

Can the eigenvector be zero?

Theorem 2.1.2.

Any square matrix has exactly n eigenvectors (some can be complex).

Remark 2.1.3.

Eigenvalues for the corresponding right and left eigenvectors coincide

Examples 2.1.1.

1) Compute eigenvalues for the matrices

(1.a) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

(1.b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2) Check if the vector $(1,1)$ is a left eigenvector for the matrix:

(2.c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3) Check if the vector $(1,0)$ is a left eigenvector for the same matrix

2.2. Extension of the notion of a matrix to higher dimensions

Sometimes a square matrix is called a **second order tensor**, if we consider a three-dimensional table that is a set with 3 indices, then we call such a table a tensor of the third order:

(15)

$$\mathbf{B} = \begin{pmatrix} b_{111}; \ b_{112}; \ b_{113}; b_{121}; \ b_{122}; \ b_{123}; \ b_{131}; \ b_{132}; \ b_{133} \\ b_{211}; \ b_{212}; \ b_{213}; b_{221}; \ b_{222}; \ b_{223}; b_{221}; \ b_{222}; \ b_{223} \\ b_{311}; \ b_{312}; \ b_{313}; b_{321}; \ b_{322}; \ b_{323}; b_{321}; \ b_{322}; \ b_{323} \end{pmatrix}$$

or

$$\mathbf{B} = \left(b_{ijk}\right)$$

Similarly, we can define tensors of the **fourth order**:

(16)

$$\mathbf{B} = \left(b_{1111}; \ ....; \ b_{3333}\right)$$

or

$$\mathbf{B} = \left(b_{ijkl}\right)$$

2.3. Functions of a matrix

Definition 2.3.1.

Let f be an analytic function of a real at $x \in (-\infty; \infty)$ and let this function be represented by the following (absolutely convergent) Taylor series:

(17) $$f(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$$

then the function $f(\mathbf{A})$, where $\mathbf{A}$ is an arbitrary square matrix is defined by

(18) $$f(\mathbf{A}) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \mathbf{A}^k$$

Definition 2.3.2.

To evaluate the zero order term in expression (18) we need the following definition

(19) $$\mathbf{A}^0 \equiv \mathbf{I}$$

which can be applied to any square matrix.

Examples 2.3.1.

According to the definitions for some analytic functions represented by the series introduced in the Part I, the following functions of a matrix are well defined for any square matrix

(20)
$$\exp(\mathbf{A}) = \frac{1}{0!}\mathbf{I} + \frac{1}{1!}\mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \ldots$$
$$= \sum_{k=1} \frac{1}{k!}\mathbf{A}^k$$

(21)
$$\sin \mathbf{A} = \frac{1}{1!}\mathbf{A} - \frac{1}{3!}\mathbf{A}^3 + \frac{1}{5!}\mathbf{A}^5 - \ldots$$

(22)
$$\cos \mathbf{A} = \frac{1}{0!}\mathbf{I} - \frac{1}{2!}\mathbf{A}^2 + \frac{1}{4!}\mathbf{A}^4 - \ldots$$

(23)
$$\sinh \mathbf{A} = \frac{1}{1!}\mathbf{A} + \frac{1}{3!}\mathbf{A}^3 + \frac{1}{5!}\mathbf{A}^5 + \ldots$$

(24)
$$\cosh \mathbf{A} = \frac{1}{0!}\mathbf{I} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{4!}\mathbf{A}^4 + \ldots$$

Remark 2.3.1.

The following should be remembered:

$$(25) \qquad f(\mathbf{A}) \neq \begin{pmatrix} f(a_{11}) & \dots & f(a_{1n}) \\ \\ f(a_{n1}) & \dots & f(a_{nn}) \end{pmatrix}$$

Remark 2.3.1.

It is interesting to note that the other definition for the hyperbolic function of a scalar argument

$$(26) \sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

can also be applied due to definition (20).

Remark 2.3.2.

Similarly to the preceding definitions a polynomial of an arbitrary square matrix is well defined

Scholium 2.3.1.

Sometimes functions are not analytic everywhere, then there can appear terms containing negative powers in the corresponding power series, for example

$$\cot(x) \equiv \operatorname{cotan}(x) = x^{-1} - \frac{1}{3}x - \frac{1}{45}x^3 - \dots \tag{27}$$

in such a case definition for the function of a matrix can be applied only to the invertible square matrices, for which we have

$$\cot(\mathbf{A}) \equiv \operatorname{cotan}(\mathbf{A}) = \mathbf{A}^{-1} - \frac{1}{3}\mathbf{A} - \frac{1}{45}\mathbf{A}^3 - \dots \tag{28}$$

Remark 2.3.3.

The problem of defining the square root of a matrix $\sqrt{\mathbf{A}}$, or more generally, any non-integer power of a matrix $\mathbf{A}^{\alpha}$ will be defined for symmetric matrices or Hermitian matrices. The corresponding definition will be introduced later on.

2.4. Classification of square matrices

Type of a matrix	Real matrix	Complex matrix
Transposed	$\mathbf{A}^t$	
Hermitian Conjugate	–	$\mathbf{A}^* = \overline{\mathbf{A}^t}$
Normal	$\mathbf{A}^t \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{A}^t$	$\mathbf{A}^* \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{A}^*$
Symmetric	$\mathbf{A}^t = \mathbf{A}$	
Hermitian	–	$\mathbf{A}^* = \mathbf{A}$
Orthogonal	$\mathbf{Q}^t = \mathbf{Q}^{-1}$	–
Unitary matrix	–	$\mathbf{U}^* = \mathbf{U}^{-1}$ **!!!**
Simple matrix	Having n linearly independent eigenvectors (possibly not mutually orthogonal)	
Nilpotent (of order n)	$\mathbf{A}^n = \mathbf{0}$	

Examples 2.4.1.

Hermitian conjugate

(29) $$\mathbf{A} = \begin{pmatrix} 4+i & -5+2i \\ 3 & 15 \end{pmatrix} \Rightarrow \mathbf{A}^* = \begin{pmatrix} 4-i & 3 \\ -5-2i & 15 \end{pmatrix}$$

Symmetric

(30) $$\begin{pmatrix} 1 & -2 & 5 \\ -2 & 0 & -7 \\ 5 & -7 & 15 \end{pmatrix}$$

Hermitian

$$(31) \qquad \begin{pmatrix} 2 & 1+i & -3 \\ 1-i & 5 & -2i \\ -3 & 2i & 0 \end{pmatrix}$$

Orthogonal

$$(32) \qquad \mathbf{Q} = \begin{pmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{pmatrix}$$

Unitary

$$(33) \qquad \mathbf{U} = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}$$

Nilpotent (of order 3)

$$(34) \qquad \mathbf{A} = \begin{pmatrix} 0 & 15 & 0 \\ 0 & 0 & 15 \\ 0 & 0 & 0 \end{pmatrix}$$

Relations between matrix classes real symmetric

$$(\text{simple}) \supset (\text{normal}) \supset (\text{Hermitian}) \supset (\text{real symmetric})$$

2.5. Properties of matrices

Theorem 2.5.1.

(A) Any normal matrix $\mathbf{A}$ can be reduced by a unitary transformation $\mathbf{H}$ to a diagonal form $\mathbf{D}$:

(35) $$\mathbf{H}^* \cdot \mathbf{A} \cdot \mathbf{H} = \mathbf{D}.$$

(B) If a given matrix $\mathbf{A}$ can be reduced to the diagonal form by a non-degenerate *unitary* transformation, then such a matrix is normal.

(C) Any normal matrix of the order n has n linearly independent **mutually orthogonal** eigenvectors (may be complex)

Definition 2.5.1.

Matrices $\mathbf{A}$ and $\mathbf{B}$ are called similar if there exists a non-degenerate transformation $\mathbf{W}$, such that

(36) $$\mathbf{A} = \mathbf{W}^{-1} \cdot \mathbf{B} \cdot \mathbf{W}$$

Theorem 2.5.2.

Matrices $\mathbf{A}$ and $\mathbf{B}$ commute with each other, i.e.

(37) $$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

if and only if they are similar.

Theorem 2.5.3.

(A) Any symmetric real(!) matrix $\mathbf{A}$ can be reduced by an orthogonal transformation $\mathbf{Q}$ to a diagonal form $\mathbf{D}$:

(38) $$\mathbf{Q}^t \cdot \mathbf{A} \cdot \mathbf{Q} = \mathbf{D}.$$

(B) If a given matrix $\mathbf{A}$ can be reduced to the diagonal form by an orthogonal transformation, then such a matrix is real symmetric.

(C) Any symmetric real matrix of the order n has n linearly independent and mutually orthogonal real eigenvectors.

Theorem 2.5.4.

(A) Any Hermitian matrix $\mathbf{A}$ can be reduced by a unitary transformation $\mathbf{H}$ to a diagonal form $\mathbf{D}$ (with real numbers on the main diagonal):

(39) $$\mathbf{H}^* \cdot \mathbf{A} \cdot \mathbf{H} = \mathbf{D}.$$

(B) If a given matrix $\mathbf{A}$ can be reduced to the diagonal form by a unitary transformation, and the diagonal matrix is real, then such a matrix is Hermitian.

(C) Any Hermitian matrix of the order n has n linearly independent and mutually orthogonal eigenvectors.

Remark 2.5.1.

If a matrix is not normal one, then

(A) It cannot be reduced to the diagonal form by a suitable unitary transformation (35).

(B) It does not have a complete set of the mutually orthogonal eigenvectors

Question 2.5.1.

What is the diagonal structure (this is called the Jordan normal form) of an arbitrary matrix, not necessary normal one?

Answer:

It contains, along with possibly other eigenvalues, the Jordan block(s) on the main diagonal

$$\mathbf{D} = \begin{pmatrix} \overline{d_1 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad} & & & & \\ & d_2 & & & \\ & & \mathbf{J}_2 & & \\ & & & \mathbf{J}_3 & \\ & & & & \mathbf{J}_7 \end{pmatrix} \tag{40}$$

where

$$\mathbf{J}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \mathbf{J}_3 = \begin{pmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix}, \ldots.. \tag{41}$$

Remark 2.5.2.

Any Jordan block can be represented by the sum of the unit matrix $\mathbf{I}$, multiplied by a corresponding eigenvalue, and a nilpotent matrix:

(42) $$\mathbf{J}_2 \equiv \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where

(43) $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is a nilpotent matrix.

Question 2.5.1.

Is there a simple procedure to see, whether the given matrix contains the Jordan blocks after reducing to the Jordan normal form?

Answer:

No, only after reducing it to the (quasi) diagonal form (the Jordan normal form).

Example 2.5.1.

Check, if matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a normal matrix

Marie Ennemond Camille Jordan

1838 - 1922

Place of birth:

2.6. Non-integer powers of normal matrices

Definition 2.6.1.

Let $\mathbf{A}$ be a normal matrix and α be a non-integer power, then

(44) $$\mathbf{A}^{\alpha} = \mathbf{H}^{*} \cdot \mathbf{D}^{\alpha} \cdot \mathbf{H}$$

Example 2.6.1.

Verify definition (44) at $\alpha = 1/2$

1. By the definition

(45) $$\mathbf{A}^{1/2} \equiv \mathbf{H}^{*} \cdot \mathbf{D}^{1/2} \cdot \mathbf{H}$$

2. Consider $\left(\mathbf{A}^{1/2}\right)^{2}$:

$$\left(\mathbf{A}^{1/2}\right)^{2} \equiv \left(\mathbf{H}^{*} \cdot \mathbf{D}^{1/2} \cdot \mathbf{H}\right) \cdot \left(\mathbf{H}^{*} \cdot \mathbf{D}^{1/2} \cdot \mathbf{H}\right) =$$

(46) $$= \mathbf{H}^{*} \cdot \mathbf{D}^{1/2} \cdot \mathbf{H} \cdot \mathbf{H}^{*} \cdot \mathbf{D}^{1/2} \cdot \mathbf{H} =$$

$$= \mathbf{H}^{*} \cdot \mathbf{D} \cdot \mathbf{H} = \mathbf{A}$$

Question 2.6.1.

What is $\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}^{1/2}$?

III. Application of matrix algebra to solving systems of the second order differential equations

3.1. Basic definitions

Definition 3.1.1.

Let a system of the linear ordinary differential equations (with homogeneous coefficients) be of the type:

(1) $$\mathbf{M}\cdot\ddot{\mathbf{x}}+\mathbf{C}\cdot\dot{\mathbf{x}}+\mathbf{K}\cdot\mathbf{x}=\mathbf{f}(t)$$

where

(2) $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are symmetric matrices of the order n,

(3) $\mathbf{x},\mathbf{f}$ are n-dimensional time-dependent vectors.

Remark 3.1.1.

A. Quite often such systems arise at the analyses of different vibration processes of systems with discrete masses.

B. Higher-order (than the second-order) equations cannot arise in the theory of vibrations of systems with discrete masses (explain, why?).

C. Sometimes, matrices $\mathbf{C},\mathbf{K}$ can depend upon $\dot{\mathbf{x}}$ or(and) $\mathbf{x}$, in these cases the equation (1) becomes non-linear one.

3.2. The main stages of setting up and solving systems of the second-order ODE (ordinary differential equations) with constant coefficients

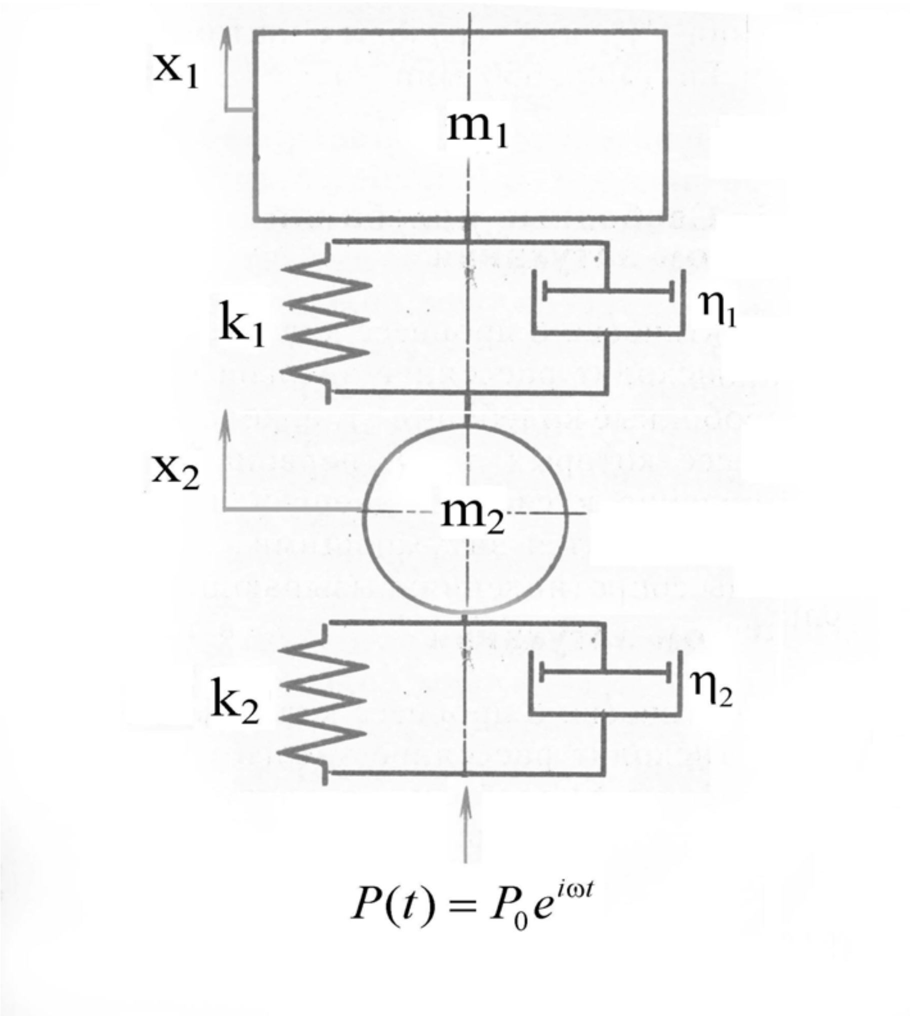
1. **Physical stage**. Setting up a system of the corresponding differential equations
2. **Matrix stage**.
   (a) Regrouping equations and obtaining a three matrix representation (1) containing n-dim matrices and vectors
   (b) Obtaining a $2n$-dimensional matrix representation, and analyzing its structure
3. **Free vibration stage (General solution stage)**. Constructing the general solution for the free-vibration problem
   (a) Euler's exponential representation
   (b) Constructing the general solution
   (c) Satisfying the initial conditions
4. **Stationary vibration stage.** Constructing a partial solution for a harmonic loading
5. **Transient response analysis.** Constructing the partial solution satisfying the initial conditions.

May not be needed

3.3. Physical stage
Setting up a system
of the second-order differential equations

Example 3.3.1. Two mass system with dampers

Let the two-mass system with dampers be as follows:



Remark 3.3.1.

To adjust the notations, henceforce we will denote the viscosity coefficients η_k as c_k.

The corresponding equations of motion are as follows:

$$(4)\quad \begin{cases} m_1\ddot{x}_1 + c_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) = 0 \\ m_2\ddot{x}_2 - c_1(\dot{x}_1 - \dot{x}_2) - k_1(x_1 - x_2) + c_2\dot{x}_2 + k_2 x_2 = P_0 e^{i\omega t} \end{cases}$$

Remark 3.3.2.

Herein we assume that m_n, c_n, k_n, where $n = 1, 2$, are some positive constants.

Remark 3.3.3.

Sometimes, coefficients c_n can be assumed to depend upon $\dot{x}_n$ (“viscosity depends upon speed”),

or

coefficients k_n can be assumed to depend upon x_n (“stiffness depends upon deformation”)

in these cases Eqs. (4) become non-linear. Solution of the non-linear differential equations is much more difficult, than linear equations.

3.4. Matrix Stage (a).
Regrouping equations and obtaining matrix representation containing n-dim matrices and vectors

Example 3.4.1.

Now, we are going to represent system (4) in a matrix form:

(5)
$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \cdot \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} c_1 & -c_1 \\ -c_1 & c_1 + c_2 \end{pmatrix} \cdot \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ Pe^{i\omega t} \end{pmatrix}$$

denoting

(6)
$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} c_1 & -c_1 \\ -c_1 & c_1 + c_2 \end{pmatrix},$$
$$\mathbf{K} = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ Pe^{i\omega t} \end{pmatrix}$$

we arrive at the desired matrix representation:

(7)
$$\mathbf{M} \cdot \ddot{\mathbf{x}} + \mathbf{C} \cdot \dot{\mathbf{x}} + \mathbf{K} \cdot \mathbf{x} = \mathbf{f}(t)$$

3.5. Matrix Stage (b). Constructing the $2n$-dimensional matrix representation

Remark 3.5.1.

A. The necessity to construct an auxiliary $2n$-dimensional matrix is due to the principle impossibility to resolve the system (1) at arbitrary symmetric matrices **M**, **C**, **K** (from physical considerations matrix **M** must be non-degenerate).

B. But, there exists a method of resolving any system of the first-order differential equations with constant coefficients of the kind

$$\dot{\mathbf{z}} = \mathbf{G} \cdot \mathbf{z} \tag{8}$$

with no restrictions imposed on matrix $\mathbf{G}$.

The main idea of this stage

is to reduce the given system of the second-order differential equations to the system of the first-order differential equations.

Reducing homogeneous system (1) (with the zero right-hand side) to the system (8)

Let

(9) $\dot{\mathbf{x}} = \mathbf{v}$,

then (homogeneous) system (1) takes the form

(10) $\mathbf{M} \cdot \dot{\mathbf{v}} + \mathbf{C} \cdot \mathbf{v} + \mathbf{K} \cdot \mathbf{x} = \mathbf{0}$

or in a following form :

(10*) $\dot{\mathbf{v}} = -\mathbf{M}^{-1} \cdot \mathbf{C} \cdot \mathbf{v} - \mathbf{M}^{-1} \cdot \mathbf{K} \cdot \mathbf{x}$

Combining (9) and (10*) we arrive at

(11) $$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1} \cdot \mathbf{K} & -\mathbf{M}^{-1} \cdot \mathbf{C} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix}$$

or in a more compact form

(12) $\dot{\mathbf{z}} = \mathbf{G} \cdot \mathbf{z}$

where

(13) $$\mathbf{G} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1} \cdot \mathbf{K} & -\mathbf{M}^{-1} \cdot \mathbf{C} \end{pmatrix}; \quad \mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix}$$

Remark 3.5.2.

If matrix $\mathbf{G}$ is a normal matrix (i.e. $\mathbf{G}^t \cdot \mathbf{G} = \mathbf{G} \cdot \mathbf{G}^t$), then it has a set of $2n$ orthonormal eigenvectors.

Example 3.5.1.

For the considered two-mass system the desired $2n$-dimensional matrix $\mathbf{G}$ takes the form

(14) $$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \Rightarrow \mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}$$

(15) $$\mathbf{C} = \begin{pmatrix} c_1 & -c_1 \\ -c_1 & c_1 + c_2 \end{pmatrix}; \mathbf{K} = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{pmatrix}$$

(16) $$\mathbf{G} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1} \cdot \mathbf{K} & -\mathbf{M}^{-1} \cdot \mathbf{C} \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{c_1}{m_1} & \frac{c_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1 + k_2}{m_2} & \frac{c_1}{m_2} & -\frac{c_1 + c_2}{m_2} \end{pmatrix}$$

(17)

and

$$\text{(18)} \qquad \mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}$$

Finally, the system of the first-order differential equations for the considered case becomes:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1}{m_1} & \dfrac{k_1}{m_1} & -\dfrac{c_1}{m_1} & \dfrac{c_1}{m_1} \\ \dfrac{k_1}{m_2} & -\dfrac{k_1+k_2}{m_2} & \dfrac{c_1}{m_2} & -\dfrac{c_1+c_2}{m_2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}$$

(19)

Remark 3.5.3.

Even for the considered relatively simple case of a two-mass system, the constructed matrix $\mathbf{G}$ can be not a normal matrix.

3.6. Free vibration stage (a)

Euler's exponential representation

Leonard Euler suggested to search a general solution of the system (8) in the form

$$\mathbf{z}(t) = \mathbf{m}\, e^{pt} \tag{20}$$

where $\mathbf{m}$ is a constant $2n$-dimensional vector, called the amplitude, and the exponential term describes possible oscillations (if p contains a non-zero imaginary part), and possible attenuation (if p contains a negative real part).

Substituting (20) into Eq. (8) yields:

$$p\mathbf{m} = \mathbf{G}\cdot\mathbf{m} \text{ or } \left(p\mathbf{I} - \mathbf{G}\right)\cdot\mathbf{m} = 0 \tag{21}$$

Thus, the problem of finding the solutions to the system (8), reduced to the problem of finding eigenvectors $\mathbf{m}$ and eigenvalues p of the matrix $\mathbf{G}$.

Remark 3.6.1.

Now, it is clear, why the question, whether matrix $\mathbf{G}$ is normal one is so important. For a normal matrix the number of eigenvectors equals to the order of a matrix, and all the eigenvectors are mutually orthogonal.

3.7. Free vibration stage (b)

Constructing the general solution

I. Matrix $\mathbf{G}$ is normal

Let $\left(p_1,\ldots,p_{2n}\right)$ be a set of all eigenvalues of the matrix $\mathbf{G}$, and $\left(\mathbf{m}_1,\ldots,\mathbf{m}_{2n}\right)$ is the corresponding set of all eigenvectors (they are mutually orthogonal due to normality of the matrix).

Remark 3.7.1.

All the eigenvectors $\left(\mathbf{m}_1,\ldots,\mathbf{m}_{2n}\right)$ are 2n-dimensional that means:

$$\mathbf{m}_k = \begin{pmatrix} m_k^1 \\ m_k^2 \\ \cdot \\ \cdot \\ m_k^{2n} \end{pmatrix} \tag{22}$$

where numbers $m_k^1,\ m_k^2,\ldots,m_k^{2n}$ are generally complex.

Theorem 3.7.1.

The general solution for the problem (8) has the form:

$$\mathbf{z}(t) = \sum_{k=1}^{2n} C_k \mathbf{m}_k e^{p_k t} \qquad (22)$$

Scholium 3.7.1.

Thus, for a normal matrix $\mathbf{G}$, there exists $2n$ eigenvalues (called the **natural frequencies** or spectral values), and $2n$ different and mutually orthogonal eigenvectors (called **natural modes**)

Remark 3.7.2.

Not all the natural frequencies should be different, but some may coincide (this corresponds to appearing multiple roots in the corresponding characteristic equation),

but

all the eigenvectors (natural modes) are necessary different.

Remark 3.7.3.

Coincidence of the natural frequencies means that at the coinciding frequency, different modes of vibration can occur.

II. Not-normal matrix **G**

For a not-normal matrix **G** the number of its eigenvalues can be less than the dimension of the matrix, and being reduced to the Jordan normal form it may contain Jordan blocks.

Theorem 3.7.4.

For a not-normal matrix having the Jordan blocks (non-simple matrix) the general solution of the problem (8), has the following form

$$\mathbf{z}(t) = \sum_{k=1}^{l_1} C_k \mathbf{m}_k e^{p_k t} +$$

$$\sum_{k=l_1}^{l_2} \mathbf{m}_k (C_k + C'_k t) e^{p_k t} + \sum_{k=l_1}^{l_2} C'_k \mathbf{m}'_k e^{p_k t} +$$

$$\sum_{k=l_2}^{l_3} \mathbf{m}_k (C_k + C'_k t + \frac{1}{2} C''_k t^2) e^{p_k t} +$$

$$\sum_{k=l_2}^{l_3} (C'_k + C''_k t) \mathbf{m}'_k e^{p_k t} + \sum_{k=l_2}^{l_3} C''_k \mathbf{m}''_k e^{p_k t} +$$

$$\ldots\ldots\ldots$$

(23)

3.8. Free vibration stage (c)

Satisfying the initial conditions (Normal or simple matrix $\mathbf{G}$)

Let the general solution (22) satisfy the initial conditions at some time t_0 (usually t_0 is taken to be zero, but that is not necessary):

$$\text{(24)} \qquad \begin{cases} \mathbf{u}(t_0) = \mathbf{u}_0 \\ \dot{\mathbf{u}}(t_0) \equiv \mathbf{v}(t_0) = \mathbf{v}_0 \end{cases}$$

or in the form

$$\text{(25)} \qquad \mathbf{z}(t_0) = \mathbf{z}_0$$

where as before

$$\text{(26)} \qquad \mathbf{z}(t) \equiv \begin{pmatrix} \mathbf{u}(t) \\ \mathbf{v}(t) \end{pmatrix}; \quad \mathbf{z}_0 = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \end{pmatrix}$$

Recalling that the general representation for the vector $\mathbf{z}$ is given by (22), we arrive at the following system for finding the unknown coefficients C_k:

$$\text{(27)} \qquad \sum_{k=1}^{2n} C_k \mathbf{z}_k(t_0) = \mathbf{z}_0$$

Theorem 3.8.1.

For any normal (and not degenerate) matrix $\mathbf{G}$ and any $2n$-dimensional vector $\mathbf{z}_0$ the system (27) is uniquely solvable.

Question 3.8.1.

Why for a normal and non-degenerate matrix $\mathbf{G}$, the system (27) is uniquely solvable?

Remark 3.8.1.

A. For a non-normal matrix $\mathbf{G}$ the initial value problem (27) is also uniquely solvable, but the construction of the solution can be more elaborate.

B. Eigenvalues of the matrix $\mathbf{G}$ (natural frequencies) satisfy the following polynomial equation of the $2n$-order:

$$\det\left(p^2\mathbf{M} + p\mathbf{C} + \mathbf{K}\right) = 0 \tag{28}$$

This equation is derived by substituting the reduced Euler's representation (for the n-dimensional amplitude vector $\mathbf{m}'$) into the initial differential equation (1).

Theorem 3.8.2.

All the eigenvalues p have non-negative real part $\mathrm{Re}(p) \le 0$, provided matrices $\mathbf{C}, \mathbf{K}$ are positive semi-definite and matrix $\mathbf{M}$ is positive definite.

Proof.

Consider Eq. (20) with a $2n$-dimensional eigenvector $\mathbf{m}$ representing in a form

(29) $$\mathbf{m} = \begin{pmatrix} \mathbf{m}' \\ \mathbf{m}'' \end{pmatrix}$$

where $\mathbf{m}'$, $\mathbf{m}''$ are n-dimensional vectors (due to (13) $\mathbf{m}'$ stands for displacements and $\mathbf{m}''$ stands for speeds). Substituting vector $\mathbf{m}$ in the form (29) into Eq. (20), we arrive at

(30) $$p^2\mathbf{M} \cdot \mathbf{m}' + p\mathbf{C} \cdot \mathbf{m}' + \mathbf{K} \cdot \mathbf{m}' = 0$$

no we multiply both sides by the same vector $\mathbf{m}'$, and find the equation for p:

(31) $$p^2\mathbf{m}' \cdot \mathbf{M} \cdot \mathbf{m}' + p\mathbf{m}' \cdot \mathbf{C} \cdot \mathbf{m}' + \mathbf{m}' \cdot \mathbf{K} \cdot \mathbf{m}' = 0$$

roots of the latter equation are

$$p = -\frac{\mathbf{m}'\cdot\mathbf{C}\cdot\mathbf{m}'}{2\mathbf{m}'\cdot\mathbf{M}\cdot\mathbf{m}'} \pm \sqrt{\left(\frac{\mathbf{m}'\cdot\mathbf{C}\cdot\mathbf{m}'}{2\mathbf{m}'\cdot\mathbf{M}\cdot\mathbf{m}'}\right)^2 - \frac{\mathbf{m}'\cdot\mathbf{K}\cdot\mathbf{m}'}{\mathbf{m}'\cdot\mathbf{M}\cdot\mathbf{m}'}}$$

(32)

The condition of the theorem ensures both of the roots to have a non-positive real part.

Scholium 3.8.1.

The condition $\mathrm{Re}(p) \leq 0$ is of high importance for the whole theory of vibrations. It states that nevertheless of complexity of the vibrating system, all the vibrations can be either attenuating with time, or constantly oscillating, with no exponential (or any other) growth. Thus, **no unbounded motions can exist**.

Question 3.8.2.

A. Why matrix $\mathbf{M}$ is positive definite one, and matrices $\mathbf{C}$, $\mathbf{K}$ are positive semi-definite?

B. What does semi-definiteness mean?

C. What is a necessary and sufficient condition for purely oscillating motion without attenuation? (consider expression for the roots (32))

Addendum (1) to the free vibration stage

Another useful representation for the general solution based on $2n$-dimensional formalism

Herein, we assume that matrix $\mathbf{G}$ has simple structure. Now, we shall regard matrix

$$e^{\mathbf{G}t} \tag{A1.1}$$

Theorem A1.

Matrix (A1) is a fundamental matrix: each column of this matrix is the corresponding (right) eigenvector

Proof.

Substituting matrix (A1.1) into Eq. (12), yields the following identity:

$$\mathbf{G} \cdot e^{\mathbf{G}t} = \mathbf{G} \cdot e^{\mathbf{G}t} \tag{A1.2}$$

It can also be proved that each column corresponds to an eigenvector.

Satisfying initial conditions (27) by using matrix (A1.1) yields:

$$e^{\mathbf{G}t_0} \cdot \vec{C} = \mathbf{z}_0 \quad \Rightarrow \quad \vec{C} = e^{-\mathbf{G}t_0} \cdot \mathbf{z}_0 \tag{A1.3}$$

Now, from the last equation in (A1.3), we can represent the solution, satisfying initial conditions (27) in the following elegant form:

$$\text{(A1.4)} \quad \mathbf{z}(t) = e^{\mathbf{G}(t-t_0)} \cdot \mathbf{z}_0$$

Remark A1.

A. The solution of the homogeneous equation (12) in the form (A1.4) is valid for any non-degenerate matrix $\mathbf{G}$ having simple structure (without Jordan blocks).

B. If matrix $\mathbf{G}$ is not simple (i.e. it has the Jordan blocks, then the solution of the stationary vibration problem in the form (A1.4) cannot be obtained, and a more elaborate method should be used.

3.9. Eigenproblem for a two-mass system

Constructing the general solution

Example 3.9.1.

Let for the considered in Sec. 3.3 the dynamical two mass-system, matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ be as follows:

$$\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \Rightarrow \mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \tag{33}$$

$$\mathbf{C} = \begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix}; \quad \mathbf{K} = \begin{pmatrix} 2 & -2 \\ -2 & 7 \end{pmatrix} \tag{34}$$

then

$$\mathbf{G} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & -\frac{7}{3} & -\frac{1}{3} & -\frac{5}{3} \end{pmatrix} \tag{35}$$

Remark 3.9.1.

Direct verification reveals that matrix (35) is not a normal one, but still it has $2n$ eigenvectors (linear independent, but not mutually orthogonal)

Eigenvalues or **natural frequencies** of the matrix $\mathbf{G}$ (obtained numerically):

$$
\begin{array}{c}
-0.199+0.764i \\
-0.199-0.764i \\
-0.884+1.375i \\
-0.884-1.375i
\end{array}
\tag{36}
$$

Remark 3.9.2.

A. Sign "–" at the real part of the eigenvectors means attenuating with time

B. Complex structure of the eigenvalues means that oscillations are not constant with time.

Eigenvectors $\mathbf{m}_k$ of the matrix $\mathbf{G}$ (obtained numerically):

$$
\begin{aligned}
&\left\{\begin{array}{c}
-0.8262194743-0.1211711982i \\
-0.3108864455+0.01249224851i \\
0.257022269-0.6073126767i \\
0.05231987968-0.2400763171i
\end{array}\right\} \leftrightarrow -0.1990015834+0.7642351156i, \\
&\left\{\begin{array}{c}
-0.8262194743+0.1211711982i \\
-0.3108864455-0.01249224851i \\
0.257022269+0.6073126767i \\
0.05231987968+0.2400763171i
\end{array}\right\} \leftrightarrow -0.1990015834-0.7642351156i, \\
&\left\{\begin{array}{c}
-0.010849974+0.4854766403i \\
-0.3474453165-0.9411900004i \\
0.6770797607-0.4144047089i \\
-0.9867909244+1.310028877i
\end{array}\right\} \leftrightarrow -0.8843317522-1.374906087i, \\
&\left\{\begin{array}{c}
-0.010849974-0.4854766403i \\
-0.3474453165+0.9411900004i \\
0.6770797607+0.4144047089i \\
-0.9867909244-1.310028877i
\end{array}\right\} \leftrightarrow -0.8843317522+1.374906087i
\end{aligned}
\tag{37}
$$

Remark 3.9.3.

At this stage the complex structure of the eigenvalues should be retained.

Satisfying boundary conditions by solving Eq. (27)

Matrix consisting of the eigenvectors:

$$\begin{pmatrix} -0.8262194743-0.1211711982i & -0.8262194743+0.1211711982i & -0.010849974+0.4854766403i & -0.010849974-0.4854766403i \\ -0.3108864455+0.01249224851i & -0.3108864455-0.01249224851i & -0.3474453165-0.9411900004i & -0.3474453165+0.9411900004i \\ 0.257022269-0.6073126767i & 0.257022269+0.6073126767i & 0.6770797607-0.4144047089i & 0.6770797607+0.4144047089i \\ 0.05231987968-0.2400763171i & 0.05231987968+0.2400763171i & -0.9867909244+1.310028877i & -0.9867909244-1.310028877i \end{pmatrix}$$

The right-hand side (initial conditions) :

$$\mathbf{z}_0 \equiv \begin{pmatrix} x_1^0 \\ x_2^0 \\ v_1^0 \\ v_2^0 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 0 \\ 10 \end{pmatrix} \qquad (38)$$

Question 3.9.1.

How to interpret the last vector in the right-hand side of (38), what the first two components $(5,7)$ and the last two $(0,10)$ represent?

Solving system (27) gives values for the unknown coefficients C_k, $k = 1,2,3,4$:

$$\begin{matrix} -2.671680868+6.107165094i \\ -2.671680868-6.107165094i \\ -5.097474057+1.035507569i \\ -5.097474054-1.035507569i \end{matrix} \qquad (39)$$

Remark 3.9.4.

And at this stage the complex structure of the coefficients should be retained.

Now, we are able to write the complete solution in the form (22) for the regarded free-vibration problem by multiplying eigenvectors (37) (with the corresponding exponents) by coefficients (39)

$$(40) \qquad \mathbf{z}(t) = \sum_{k=1}^{4} C_k \mathbf{m}_k e^{p_k t} =$$

$$\begin{pmatrix} (2.947-4.722i)e^{p_1 t} \\ (0.7543-1.932i)e^{p_1 t} \\ (3.022+3.192i)e^{p_1 t} \\ (1.326+0.9609i)e^{p_1 t} \end{pmatrix} + \begin{pmatrix} (2.947+4.722i)e^{p_2 t} \\ (0.7543+1.932i)e^{p_2 t} \\ (3.022-3.192i)e^{p_2 t} \\ (1.326-0.9609i)e^{p_2 t} \end{pmatrix} + \begin{pmatrix} (-0.4474-2.486i)e^{p_3 t} \\ (2.746+4.438i)e^{p_3 t} \\ (-3.022+2.814i)e^{p_3 t} \\ (3.674-7.700i)e^{p_3 t} \end{pmatrix} + \begin{pmatrix} (-0.4474+2.486i)e^{p_4 t} \\ (2.746-4.438i)e^{p_4 t} \\ (-3.022-2.814i)e^{p_4 t} \\ (3.674+7.700i)e^{p_4 t} \end{pmatrix}$$

Remark 3.9.5.

Even at this stage the complex structure of the general solution should be retained, as the exponents can also be complex, and they are complex for the considered case; see eigenvalues (36).

Question 3.9.2.

A. What are the first two and the last two elements in each column?

B. How to evaluate the exponent $e^{p_1 t}$?

Thus, the general solution satisfying the initial conditions is constructed. Only now, we should retain either real or imaginary part of the solution.

3.10. Stationary vibration stage

Constructing a partial solution for harmonic loading

Basic assumptions.

A. Let a harmonic loading applied to the corresponding masses be

$$\mathbf{f}(t) = \mathbf{f}_0 e^{i\omega t} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} e^{i\omega t} \qquad (41)$$

where f_k are the external force amplitudes, and ω is the forcing frequency (ω is real, and f_k can be complex)

B. From now it is assumed that the loading frequency ω does not coincide with any of the eigenvalues (natural frequencies) p_k, $k = 1, \ldots, 2n$.

Remark 3.10.1.

If the parameter $i\omega$ coincides with any one of the natural frequencies, then the solution does not exist, this is called the resonance.

Constructing the solution.
There can be two variants of constructing the solution:

I. ON THE BASIS OF n-DIMENSIONAL MATRICES:

A. We are looking the solution in the form:

(42) $$\mathbf{x}(t) = \mathbf{x}_0 e^{i\omega t}$$

where $\mathbf{x}_0$ is the unknown n-dimensional amplitude vector.

B. Obtaining vector $\mathbf{x}_0$ is done by substituting both (41) and (42) into Eq. (1), arriving at:

(43) $$\left(-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right)\cdot\mathbf{x}_0 = \mathbf{f}_0$$

and since the matrix $\left(-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right)$ is invertible, the final solution takes the form:

(44) $$\mathbf{x}_0 = \left(-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right)^{-1}\cdot\mathbf{f}_0$$

Remark 3.10.2.
The vector $\mathbf{x}_0$ can be complex, and its complex structure should be retained up to multiplication by the complex exponent $e^{i\omega t}$.

II. ON THE BASIS OF $2n$-DIMENSIONAL MATRIX $\mathbf{G}$:

A. We are looking the solution in the form:

(45) $$\mathbf{z}(t) = \mathbf{z}_0 e^{i\omega t}$$

where $\mathbf{z}_0$ is the unknown $2n$-dimensional amplitude vector (the first n elements of which are amplitudes of displacements, and the last n are the amplitudes of speeds). This vector should satisfy the equation with the external forces:

(12′) $$\dot{\mathbf{z}}(t) = \mathbf{G} \cdot \mathbf{z}(t) + \mathbf{w}(t)$$

B. Now, we construct an auxiliary $2n$-dimensional loading vector:

(46) $$\mathbf{w}(t) = \mathbf{w}_0 e^{i\omega t} \equiv \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \cdot \mathbf{f}_0 \end{pmatrix} e^{i\omega t}$$

C. Obtaining $\mathbf{z}_0$. since $i\omega$ does not coincide with any of the natural frequencies, we can invert matrix $(i\omega\mathbf{I} - \mathbf{G})$ and write the solution for $\mathbf{z}_0$ in the form:

(47) $$\mathbf{z}_0 = (i\omega\mathbf{I} - \mathbf{G})^{-1} \cdot \mathbf{w}_0$$

Remark 3.10.3.

In Eq. (47) matrix $\mathbf{I}$ is the $2n$-dimensional identity matrix.

Remark 3.10.4.

Both of the considered methods can be applied to systems of harmonic loadings containing m different loadings corresponding to m different frequencies (number of different loadings m can be arbitrary), namely

$$\mathbf{f}(t) = \sum_{k=1}^{m} \mathbf{f}_k e^{i\omega_k t} \tag{48}$$

where $\mathbf{f}_k$ are n-dimensional vectors, representing force amplitudes, applied to the considered system with n-masses. Provided neither of frequencies ω_k coincide with the natural frequencies (eigenvalues).

In view of (48), we can obtain the following expression for displacements of vibrating masses, which generalizes method, based on the n-dimensional matrix approach; see formula (44):

$$\mathbf{x}(t) = \sum_{k=1}^{m} \mathbf{x}_k e^{i\omega_k t} = \sum_{k=1}^{m} \left(-\omega_k{}^2\mathbf{M} + i\omega_k\mathbf{C} + \mathbf{K}\right)^{-1} \cdot \mathbf{f}_k e^{i\omega_k t} \tag{49}$$

Generalization of the $2n$-dimensional method is similar.

Remark 3.10.5.

Both of the considered methods can be applied to systems of harmonic loadings with exponential attenuation or even exponential growth:

(50) $$\mathbf{f}(t) = \sum_{k=1}^{m} \mathbf{f}_k e^{(\alpha_k + i\omega_k)t}$$

where $\alpha_k < 0$ corresponds to attenuation, and $\alpha_k > 0$ corresponds to exponential growth. The solution is available, provided neither of $\alpha_k + i\omega_k$ coincide with the corresponding eigenvalues.

The following formula generalizes formula (49):

$$\mathbf{x}(t) = \sum_{k=1}^{m} \mathbf{x}_k e^{(\alpha_k + i\omega_k)t} =$$

$$\sum_{k=1}^{m} \left((\alpha_k + i\omega_k)^2 \mathbf{M} + (\alpha_k + i\omega_k)\mathbf{C} + \mathbf{K} \right)^{-1} \cdot \mathbf{f}_k \times$$

$$\times e^{(\alpha_k + i\omega_k)t}$$

(51)

Addendum (2) to the free-vibration stage:
Definition and properties of the fundamental matrices

Definition of the fundamental matrix.

Fundamental matrix is a functional matrix $\mathbf{M}(t)$, when substituted in the initial differential equation (12) instead of vector $\mathbf{z}(t)$, satisfies this equation, i.e.

(A2.1) $$\dot{\mathbf{M}} = \mathbf{G} \cdot \mathbf{M}$$

Remark A2.1.

The fundamental matrix may be not a unique matrix (there can be several or even infinite different fundamental matrices)

Theorem A2.1.

Suppose that matrix $\mathbf{G}$ is a simple matrix (it does not contain any Jordan blocks), then a matrix composed of eigenvectors of the matrix $\mathbf{G}$ multiplied by the exponents with the corresponding eigenvalues in the indices, is the fundamental matrix, i.e.

(A2.2) $$\mathbf{M} = \left(\mathbf{m}_1 e^{p_1 t};...;\mathbf{m}_{2n} e^{p_{2n} t} \right)$$

where $\mathbf{m}_k$ are eigenvectors, and p_k are the corresponding eigenvalues.

Proof of the Theorem A2.1.

A. Let us multiply matrix $\mathbf{G}$ by the first column of the matrix $\mathbf{M}$, i.e. $2n$-dimensional vector $\mathbf{m}_1 e^{p_1 t}$, this yields

(A2.3) $$\mathbf{G} \cdot \mathbf{m}_1 e^{p_1 t} = p_1 \mathbf{m}_1 e^{p_1 t}$$

Performing this multiplications for other columns of the matrix $\mathbf{M}$, we get

(A2.4) $$\mathbf{G} \cdot \mathbf{M} = \left(p_1 \mathbf{m}_1 e^{p_1 t}; \ldots; p_{2n} \mathbf{m}_{2n} e^{p_{2n} t} \right)$$

B. Differentiating matrix $\mathbf{M}$, gives

(A2.5) $$\dot{\mathbf{M}} = \left(p_1 \mathbf{m}_1 e^{p_1 t}; \ldots; p_{2n} \mathbf{m}_{2n} e^{p_{2n} t} \right)$$

C. Comparing right-hand sides in (A2.4) and (A2.5), we conclude that matrix $\mathbf{M}$ satisfies the equation (A2.1), thus $\mathbf{M}$ is a fundamental matrix.

Proposition A2.1.

For a simple matrix $\mathbf{G}$, the correspondent matrix $\mathbf{M}$ is not singular ($\det \mathbf{M} \neq 0$).

Proof.

Proof is obvious, since simple matrix $\mathbf{G}$ has $2n$ linear independent eigenvectors.

Theorem A2.2.

Suppose that matrix $\mathbf{G}$ is a simple matrix (it does not contain any Jordan blocks), then matrix $e^{\mathbf{G}t}$ is the fundamental matrix, i.e.

$$\frac{d}{dt} e^{\mathbf{G}t} = \mathbf{G} \cdot e^{\mathbf{G}t} \qquad \text{(A2.6)}$$

Proof of the Theorem A2.2.

Proof is straightforward, since after differentiating the left-hand side becomes equal to the right-hand side.

Proposition A2.2.

For a simple matrix $\mathbf{G}$, matrix $e^{\mathbf{G}t}$ is not singular ($\det e^{\mathbf{G}t} \neq 0$).

Proof.

Let us reduce matrix $\mathbf{G}$ to the diagonal form:

$$\mathbf{G} = \mathbf{W}^{-1} \cdot \mathbf{D} \cdot \mathbf{W} \qquad \text{(A2.7)}$$

where $\mathbf{W}$ is an arbitrary non-singular square matrix. Now, we consider definition for the exponential matrix, and use decomposition (A2.7):

(A2.8) $$e^{\mathbf{G}t} = \sum_{k=0}^{\infty} \frac{\mathbf{G}^k t^k}{k!}$$

In the right-hand side of (A2.8) I'll use decomposition (A2.7), it gives for the k-th power:

$$\mathbf{G}^k = \underbrace{\mathbf{W}^{-1} \cdot \mathbf{D} \cdot \mathbf{W}}_{\mathbf{G}} \cdot \underbrace{\mathbf{W}^{-1} \cdot \mathbf{D} \cdot \mathbf{W}}_{\mathbf{G}} \cdot \ldots \cdot \underbrace{\mathbf{W}^{-1} \cdot \mathbf{D} \cdot \mathbf{W}}_{\mathbf{G}} =$$

$$= \mathbf{W}^{-1} \cdot \mathbf{D}^k \cdot \mathbf{W} = \mathbf{W}^{-1} \cdot \begin{pmatrix} d_1^k & & & \\ & d_2^k & & \\ & & & \\ & & & d_{2n}^k \end{pmatrix} \cdot \mathbf{W}$$

(A2.9)

By use of (A2.9), we can now represent $e^{\mathbf{G}t}$ in the form:

(A2.10) $$e^{\mathbf{G}t} = \mathbf{W}^{-1} \cdot \begin{pmatrix} e^{d_1 t} & & & \\ & e^{d_2 t} & & \\ & & & \\ & & & e^{d_{2n} t} \end{pmatrix} \cdot \mathbf{W}$$

and since the exponent function never vanishes (at any exponents either real or complex), formula (A2.10) completes the proof.

Properties of the fundamental matrices.

Proposition A2.3.

Matrices $\mathbf{G}$ and $e^{\mathbf{G}}$ commute with each other:

$$\text{(A2.11)} \qquad \mathbf{G} \cdot e^{\mathbf{G}} = e^{\mathbf{G}} \cdot \mathbf{G}$$

Proof.

The proof follows from multiplying the Taylor series representation for $e^{\mathbf{G}}$ by matrix $\mathbf{G}$ (initially from left):

$$\text{(A2.11)} \quad \mathbf{G} \cdot e^{\mathbf{G}} = \mathbf{G} \cdot \left(\sum_{k=0}^{\infty} \frac{\mathbf{G}^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{\mathbf{G}^{k+1}}{k!}$$

It is clear that the result of multiplication of $e^{\mathbf{G}}$ by $\mathbf{G}$ from right will give the same.

Corollary 1.

Matrices $\mathbf{G}$ and $e^{\mathbf{G}}$ can be reduced to the diagonal form by the same non-degenerate transformation $\mathbf{W}$.

Corollary 2.

Matrices $\mathbf{G}$ and $e^{\mathbf{G}}$ have the same set of $2n$ linearly independent eigenvectors (**but their eigenvalues differ**).

Remark A2.1.

If $\begin{pmatrix} d_1 & & \\ & \dots & \\ & & d_{2n} \end{pmatrix}$ are eigenvalues of matrix $\mathbf{G}$, then

$\begin{pmatrix} e^{d_1} & & \\ & \dots & \\ & & e^{d_{2n}} \end{pmatrix}$ are eigenvalues of the matrix $e^{\mathbf{G}}$

Corollary 3 (The most important).

Matrices $\mathbf{M}$ and $e^{\mathbf{G}t}$ coincide (possibly up to an arbitrary scalar multiplier).

Concluding remark.

Thus, instead of computing eigenvalues and eigenvectors to construct then the fundamental matrix $\mathbf{M}$, which can be rather complicated procedure especially for high dimensional problems, it is possible to compute matrix $e^{\mathbf{G}t}$, and that will give us the same fundamental matrix.

Addendum (3) to the stationary vibration stage
Inverting matrix $\left(i\omega\mathbf{I}-\mathbf{G}\right)$

Definition of the von Neumann method.

Let a harmonic loading applied to the corresponding masses has oscillation frequency ω satisfying the inequality

(A3.1) $$\omega > \left|p_k\right|, \quad k = 1,\ldots,2n$$

That means that the loading frequency is higher than any of the eigenfrequencies (natural frequencies).

Now, the inverse matrix to the matrix $\left(i\omega\mathbf{I}-\mathbf{G}\right)$ can be obtained by Neumann's series:

(A3.2) $$\left(i\omega\mathbf{I}-\mathbf{G}\right)^{-1} = (i\omega)^{-1}\left(\sum_{n=0}^{\infty}(i\omega)^{-n}\mathbf{G}^{n}\right)$$

Thus, if condition (A3.1) is satisfied, then for obtaining amplitudes $\mathbf{z}_0$ by applying (47), we do not need to invert matrix $\left(i\omega\mathbf{I}-\mathbf{G}\right)$, but perform only successive summations of matrices $(i\omega)^{-n}\mathbf{G}^{n}$.

Theorem A3.1.

Series in the right-hand side of (A3.2) converge absolutely, provided condition (A3.1) is satisfied.

3.11. Transient response stage

Part I.

Constructing the solution for non-harmonic loading by the fundamental matrix method

Basic assumption.

Let a harmonic loading applied to the corresponding masses be

$$\mathbf{f}(t) == \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix} \tag{52}$$

where $f_k(t)$ are the external forces, applied at the corresponding masses. From now we will assume, that this loading was applied at the moment t_0.

Now, the equation of forced vibrations in a $2n$-dimensional space takes the form:

(53) $$\dot{\mathbf{z}}(t) = \mathbf{G} \cdot \mathbf{z}(t) + \mathbf{w}(t),$$

where as in the harmonic loading stage $\mathbf{w}(t)$ is a modified $2n$-dimensional loading vector of the form

(54) $$\mathbf{w}(t) = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \cdot \mathbf{f}(t) \end{pmatrix}$$

To understand how we obtained this vector, it is needed to regard the initial Eq. (1) and, as before, by introducing a new vector function $\mathbf{v} = \dot{\mathbf{x}}$, to reduce Eq. (1) to the following form

(55) $$\begin{aligned} &\dot{\mathbf{x}} = \mathbf{v} \\ &\mathbf{M} \cdot \dot{\mathbf{v}} = -\mathbf{K} \cdot \mathbf{x} - \mathbf{C} \cdot \mathbf{v} + \mathbf{f}(t) \end{aligned}$$

or in the equivalent form

(55′) $$\begin{aligned} &\dot{\mathbf{x}} = \mathbf{v} \\ &\dot{\mathbf{v}} = -\mathbf{M}^{-1} \cdot \mathbf{K} \cdot \mathbf{x} - \mathbf{M}^{-1}\mathbf{C} \cdot \mathbf{v} + \mathbf{M}^{-1}\mathbf{f}(t) \end{aligned}$$

or in another equivalent form, given by (53).

Proposition 3.11.1.

The equation

$$\frac{d}{dt}\left(e^{-\mathbf{G}t}\cdot\mathbf{z}\right)=e^{-\mathbf{G}t}\cdot\mathbf{w}(t) \tag{56}$$

is equivalent to the initial Eq.(53).

Proof.

Equivalence is verified by performing differentiation in the left-hand side of (56), and recalling that matrices $\mathbf{G}$ and $e^{\mathbf{G}t}$ (and $e^{-\mathbf{G}t}$) commute with each other.

Constructing the desired solution of Eq. (56)

$$e^{-\mathbf{G}t}\cdot\mathbf{z}(t)-e^{-\mathbf{G}t_0}\cdot\mathbf{z}(t_0)=\int_{t_0}^{t}e^{-\mathbf{G}\tau}\cdot\mathbf{w}(\tau)\,d\tau \tag{57}$$

from which we obtain the desired solution for $\mathbf{z}(t)$:

$$\mathbf{z}(t)=e^{\mathbf{G}(t-t_0)}\cdot\mathbf{z}_0+e^{\mathbf{G}t}\cdot\int_{t_0}^{t}e^{-\mathbf{G}\tau}\cdot\mathbf{w}(\tau)\,d\tau \tag{58}$$

Remark 3.11.1.

It is easily verified that the constructed solution (58) satisfies at $t=t_0$ the initial condition $\mathbf{z}_0$ (since the integral in (58) for any locally integrable function $\mathbf{w}$ vanishes at $t=t_0$).

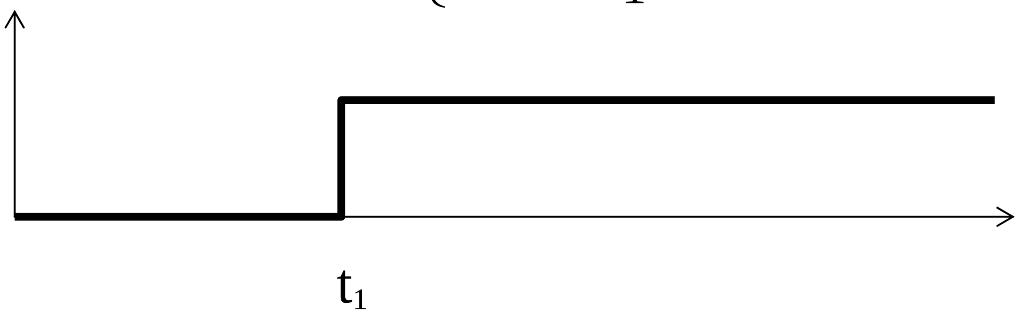
Example 3.11.1.

Heaviside loading applied at time t_1.

(59) $$\mathbf{f}(t) = \mathbf{f}_1 \, h(t - t_1)$$

where $\mathbf{f}_1$ is a constant force loading applied at masses of the considered system, and $h(t - t_1)$ is the Heaviside function:

(60) $$h(t - t_1) = \begin{cases} 0, & t < t_1 \\ 1, & t \geq t_1 \end{cases}$$



Now, according to (54) we must construct the auxiliary $2n$-dimensional loading:

(61) $$\mathbf{w}(t) = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \cdot \mathbf{f}_1 h(t - t_1) \end{pmatrix} = \mathbf{w}_1 h(t - t_1)$$

where

(62) $$\mathbf{w}_1 = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \cdot \mathbf{f}_1 \end{pmatrix}$$

substituting this loading into Eq. (58) and performing integration gives:

$$
\begin{aligned}
\mathbf{z}(t) &= e^{\mathbf{G}(t-t_0)} \cdot \mathbf{z}_0 + \\
&\quad (\mathbf{G}^{-1}) \cdot \left(e^{\mathbf{G}(t-t_1)} - \mathbf{I} \right) \cdot \mathbf{w}_1 h(t-t_1)
\end{aligned}
\tag{63}
$$

Remark 3.11.2.

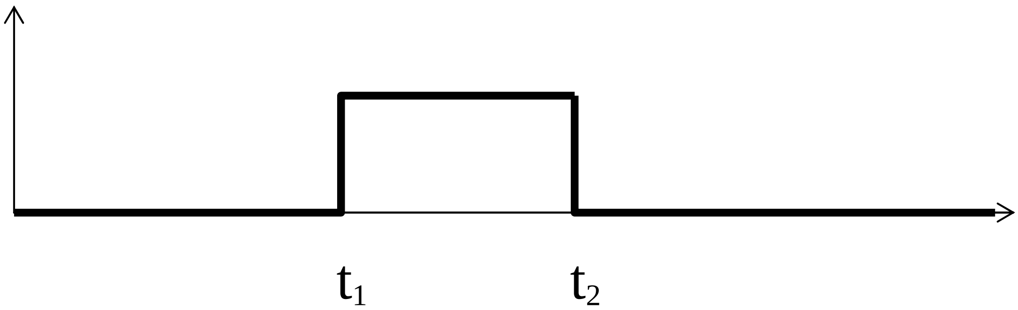
Appearing the Heaviside function in the right-hand side of (63) is because of necessity to reflect the fact that time is irreversible (event at time t_1 cannot affect earlier events).

Remark 3.11.3.

This solution is especially important, because any continuous function can be approximated with any accuracy (on a bounded interval) by a sequence of Heaviside functions, this is called the step-function approximation.

Example 3.11.2.

Step loading applied at time t_1 and stopped at t_2.



Performing direct integration by (58), or applying superposition of two previous solutions (one for $\mathbf{w}_1 h(t-t_1)$ and another for $-\mathbf{w}_1 h(t-t_2)$, we get:

$$\mathbf{z}(t) = e^{\mathbf{G}(t-t_0)} \cdot \mathbf{z}_0 +$$

$$(\mathbf{G}^{-1}) \cdot \begin{pmatrix} \left(e^{\mathbf{G}(t-t_1)} - \mathbf{I}\right) h(t-t_1) \\ -\left(e^{\mathbf{G}(t-t_2)} - \mathbf{I}\right) h(t-t_2) \end{pmatrix} \cdot \mathbf{w}_1$$

(64)

where $\mathbf{w}_1$ is defined by (62).

Remark 3.11.4.

Again, appearing Heaviside functions in the right-hand side of (64) reflects the fact that time is irreversible (events applied at times t_1 and t_2 cannot affect earlier events).

Example 3.11.3.

An impulse (δ-function) loading applied at time t_1.

Let

(65) $$\mathbf{f}(t) = \mathbf{f}_0 \delta(t - t_1)$$

where $\mathbf{f}_0$ is a force loading applied at masses of the considered system. According to (54) we must construct the auxiliary $2n$-dimensional loading:

(66) $$\mathbf{w}(t) = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \cdot \mathbf{f}_0 \delta(t - t_1) \end{pmatrix} = \mathbf{w}_0 \delta(t - t_1)$$

where $\mathbf{w}_0 = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \cdot \mathbf{f}_0 \end{pmatrix}$

substituting this loading into Eq. (58) gives:

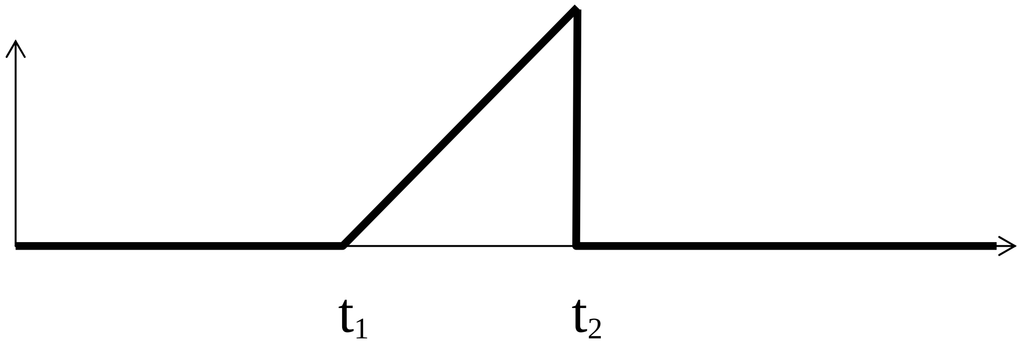
(67) $$\mathbf{z}(t) = e^{\mathbf{G}(t - t_0)} \cdot \mathbf{z}_0 + e^{\mathbf{G}(t - t_1)} \cdot \mathbf{w}_0 h(t - t_1)$$

Remark 3.11.5.

As before, Heaviside function in the right-hand side of (67) reflects the fact that time is irreversible, and event at time t_1 cannot affect earlier events.

Example 3.11.4.

A triangle loading applied at time t_1 and stopped at time t_2.



(68) $$\mathbf{f}(t) = \mathbf{f}_1 \left(h(t - t_1) - h(t - t_2) \right) (t - t_1)$$

where $\mathbf{f}_1$ is a vectorial constant corresponding to the load intensity. As before, at first we construct an auxiliary $2n$-dimensional loading $\mathbf{w}(t)$:

(69) $$\mathbf{w}(t) = \mathbf{w}_1 \left(h(t - t_1) - h(t - t_2) \right) (t - t_1)$$

where

(70) $$\mathbf{w}_1 = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \cdot \mathbf{f}_1 \end{pmatrix}$$

Then, performing direct integration by (58), we get:

$$\mathbf{z}(t) = e^{\mathbf{G}(t-t_0)} \cdot \mathbf{z}_0 + e^{\mathbf{G}t} \cdot \mathbf{B}(t) \cdot \mathbf{w}_1 \tag{71}$$

where $2n$-dimensional matrix $\mathbf{B}(t)$ is

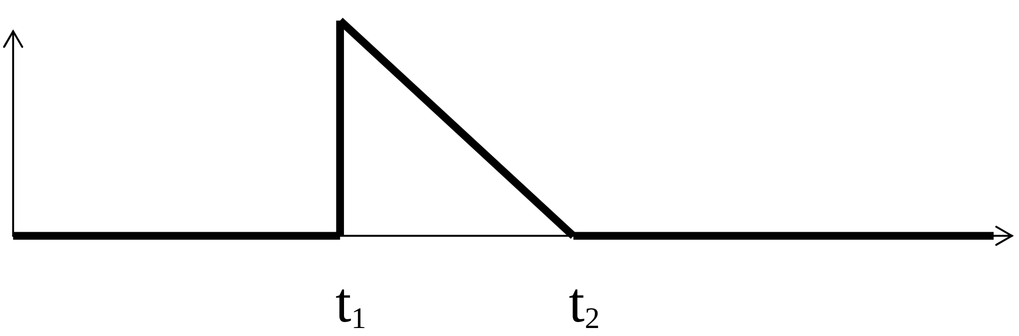
$$\mathbf{B}(t) = -\left(\mathbf{G}^{-1} \cdot e^{-\mathbf{G}t}(t-t_1) + \mathbf{G}^{-2} \cdot \left(e^{-\mathbf{G}t} - e^{-\mathbf{G}t_1}\right)\right) \times \times h(t-t_1)h(t_2-t) \tag{72}$$

Remark 3.11.5.

Analyzing expressions (69) and (72) we can observe that the unit step from example 3.11.2 can be represented in either additive or multiplicative form: $h(t-t_1) - h(t-t_2)$ or $h(t-t_1)h(t_2-t)$

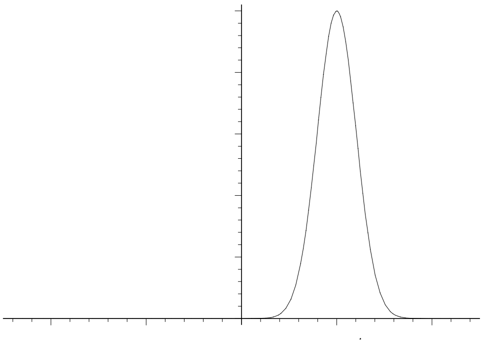
Exercise 3.11.1.

Try to construct the solution for the following triangle loading:



Example 3.11.5.

An exponential loading of the form:

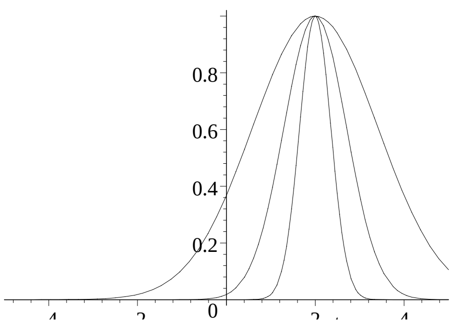


(73) $$\mathbf{f}(t) = \mathbf{f}_1\, e^{a(t_1 - t)^2}$$

where $\mathbf{f}_1$ is a vectorial constant corresponding to the load intensity (height of the peak value), parameter t_1 corresponds to the position of the extreme, and parameter a is responsible for the "width" of the graph.

Remark 3.11.6.

The following plot demonstrates influence of the parameter a on the width of the graph



The lower values for a lead to wider graphs.

Remark 3.11.7.

Strictly speaking, function (73) does not have the finite support (it means that the function does not vanish outside any finite interval), but if the parameter a is sufficiently large (>1) and time t_1 is distant from t_0 (in the plots, t_0 coincides with the origin), then in practice it can be assumed that this loading vanishes at $t < t_0$.

As before, at first we construct an auxiliary $2n$-dimensional loading $\mathbf{w}(t) = \mathbf{w}_1 e^{a(t_1 - t)^2}$, where constant $2n$-dimensional vector $\mathbf{w}_1$ have the form:

(74) $$\mathbf{w}_1 = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \cdot \mathbf{f}_1 \end{pmatrix}$$

Applying integration procedure (58), we arrive at

(75) $$\mathbf{z}(t) = e^{\mathbf{G}(t-t_0)} \cdot \mathbf{z}_0 + e^{\mathbf{G}t} \cdot \mathbf{B}(t) \cdot \mathbf{w}_1$$

where $2n$-dimensional matrix $\mathbf{B}(t)$ is

(76) $$\mathbf{B}(t) = C \operatorname{erf}\left(\frac{i}{2\sqrt{a}} \left(2a(t_1 - t)\mathbf{I} + \mathbf{G} \right) \right)$$

and C is an imaginary multiplier.

Concluding remarks for 3.11. Part I.

A. The great advantage of the considered method is in its principle ability to construct the solutions for loadings either bounded or unbounded in time domain, and without any restriction on the explored time interval (the time parameter t in (58) can be either small or large).

B. The other benefit of the method is in its straightforward nature (we do not need to expand the time dependent loading into any series to get the solution).

C. The only possible disadvantage of the method is in necessity to construct and operate with the fundamental matrix $e^{\mathbf{G}t}$.

3.11. Transient response stage

Part II.
Constructing the solution
for non-harmonic loading by the
Fourier transform method

In contrast to the preceding part, where formula (58) allowed us to construct the solution for any loading (having finite support or unbounded in time), herein it is assumed that the applied loading has finite support.

Let the analyzed time period and the applied loading are contained in the time interval $\left(t_0 ; t_1\right)$. We can expand the applied loading into Fourier series:

$$\mathbf{f}(t)=\sum_{k=-\infty}^{\infty} \mathbf{f}_k e^{i 2 \pi k t / p} \tag{77}$$

where $p=t_1-t_0$ and (generally complex) Euler coefficients are obtained by integration:

$$\mathbf{f}_k=\frac{1}{p} \int_{t_0}^{t_1} \mathbf{f}(t) e^{i 2 \pi k t / p} d t \tag{78}$$

After expanding into series (77), we can apply a method, developed in Section 3.10, and exploit either n-dimensional or $2n$-dimensional approaches for constructing the resulting solution in the form of Fourier series; see Sec.3.10 for detailed analysis.

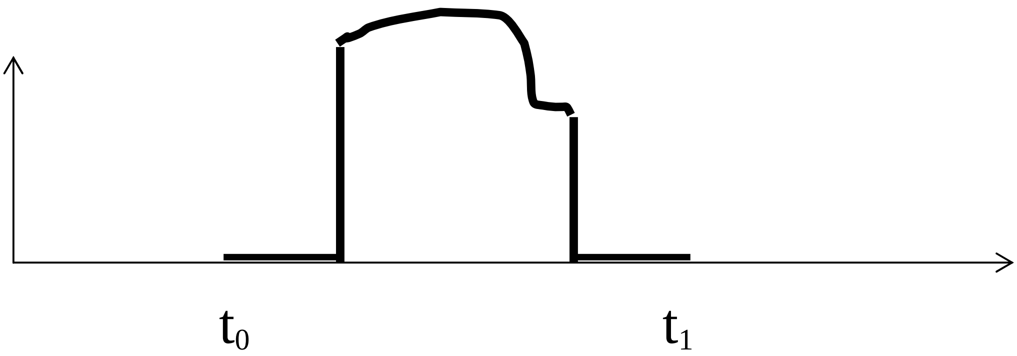
Remark 3.11.8.

At applying formulas (49) the parameters ω_k are:

(79) $$\omega_k = 2\pi kt / p$$

Remark 3.11.9.

A. The analyzed time interval $(t_0;t_1)$ can be large, than the time interval of the applied loading:



But, the interval $(t_0;t_1)$ cannot be smaller than the time interval of the applied loading, otherwise, the series expansion (77), (78) will not correspond to the applied load.

B. It can be proved that the series (77) with Euler coefficients (78) and the constructed by (48) solution converge in L^2-topology for any integrable in $(t_0;t_1)$ time-dependent vector-function $\mathbf{f}(t)$. But, the problem of summation Fourier series (sometimes it is called Fourier synthesis) with **numerically** obtained coefficients is an **ill-posed problem**. This means that if we retain large number of series, the solution instead of being convergent, becomes oscillating due to presence of numerical errors.

Concluding remarks for 3.11. Part II.

A. The obvious advantage of the Fourier expansion method is in its principle ability to avoid constructing the $e^{\mathbf{G}t}$-matrix.

B. But, it has at least two main disadvantages: (i) it cannot be applied to loadings with the unbounded time domain, and (ii) the problem of summation is an ill-posed, and special regularization methods should be applied to achieve the numerically stable results.

3.12. Problems without damping. Decoupling method

In such a case the equations of motion are

$$\mathbf{M}\cdot\ddot{\mathbf{x}}+\mathbf{K}\cdot\mathbf{x}=\mathbf{f}(t) \tag{80}$$

Free vibration stage.

Obtaining eigenvectors (natural modes) and eigenvalues (natural frequencies)

For the free vibration analysis we have two possibilities:

1. To apply the general method related to constructing an auxiliary $2n$-dimensional matrix, as it was done in the preceding sections.
2. To develop an alternative approach (known as decoupling method) without necessity to construct a $2n$- dimensional matrix.

For the regarded case equations of motion are

$$\mathbf{M}\cdot\ddot{\mathbf{x}}+\mathbf{K}\cdot\mathbf{x}=0 \tag{81}$$

The eigensolution for Eq. (81) is represented in the form of Euler's solution (20):

$$\mathbf{x}(t)=\mathbf{m}\,e^{pt} \tag{82}$$

Where $\mathbf{m}$ is a constant n-dimensional eigenvector (this vector is sometimes called as an amplitude), and p is the corresponding eigenfrequency (natural frequency).

Substituting Euler's solution (82) into Eq. (81) yields:

$$\left(p^2\,\mathbf{M}+\mathbf{K}\right)\cdot\mathbf{m}=0 \tag{83}$$

That is the Christoffel equation for the regarded problem. The Eq. (83) can be rewritten in the form:

$$\det\left(p^2\,\mathbf{M}+\mathbf{K}\right)=0 \tag{84}$$

Remark 3.12.1.

We would like to remind that both matrices $\mathbf{M}$ and $\mathbf{K}$ are symmetric, and $\mathbf{M}$ is positive definite (and in most situations is diagonal), while $\mathbf{K}$ is positive semidefinite (in real physical situations it should be even positive definite, to ensure positive potential energy)

The following theorem takes place

Theorem 3.12.1.

The eigenproblem for Eq.(81) admits n linearly independent eigenvectors, corresponding to n eigenvalues (some of the eigenvalues can coincide).

To reduce Eq. (83) to the diagonal form we rewrite Eq.(83) in the equivalent form:

$$\mathbf{M}^{1/2}\cdot\left(p^2\,\mathbf{I}+\underbrace{\mathbf{M}^{-1/2}\cdot\mathbf{K}\cdot\mathbf{M}^{-1/2}}_{\mathbf{S}}\right)\cdot\mathbf{M}^{1/2}\cdot\mathbf{m}=0$$

(85)

Now, since the matrix $\mathbf{S}=\mathbf{M}^{-1/2}\cdot\mathbf{K}\cdot\mathbf{M}^{-1/2}$ is symmetric, we can make diagonalization: $\mathbf{S}=\mathbf{Q_S}^t\cdot\mathbf{D_S}\cdot\mathbf{Q_S}$, so Eq. (85) will take place:

(86) $$\mathbf{M}^{1/2}\cdot\left(p^2\,\mathbf{I}+\mathbf{Q_S}^t\cdot\mathbf{D_S}\cdot\mathbf{Q_S}\right)\cdot\mathbf{M}^{1/2}\cdot\mathbf{m}=0$$

or, taking into account that $\mathbf{Q_S}^t\cdot\mathbf{I}\cdot\mathbf{Q_S}=\mathbf{I}$, in another form:

$$\mathbf{M}^{1/2}\cdot\mathbf{Q_S}^t\cdot\left(p^2\,\mathbf{I}+\mathbf{D_S}\right)\cdot\mathbf{Q_S}\cdot\mathbf{M}^{1/2}\cdot\mathbf{m}=0$$

(87)

The latter equation has an equivalent form analogous to Eq. (84):

(88) $$\det\left(p^2\,\mathbf{I}+\mathbf{D_S}\right)=0$$

Eqs. (87), (88) allows us to formulate

Theorem 3.12.2.

Let eigenvalues $\left(p_k\right)_{k=1,\dots,n}$ are stored in a main diagonal of a special matrix, known as the **spectral matrix**

$$(89) \qquad \Omega = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \cdots & \\ & & & p_n \end{pmatrix}$$

then

A. The spectral matrix satisfies the equation

$$(90) \qquad \Omega = \left(-\mathbf{D_S}\right)^{1/2}$$

B. All the eigenvectors (natural modes) are linearly independent columns of the matrix: $\mathbf{M}^{-1/2} \cdot \mathbf{Q_S}$.

Property
of the eigenvalues of the system without damping

All the eigenvalues are imaginary with the non-zero imaginary part (provided matrix $\mathbf{K}$ is positive definite). This flows out from considering expression (89) for the eigenvalues and the assumption that matrix $\mathbf{K}$ is positive definite.

Remark 3.12.2.

Strictly speaking, in expressions (89), (90) there should appear $2n$ eigenvalues. More correctly we should write expression (90) for the spectral matrix, as

(91) $$\mathbf{\Omega} = \pm\left(-\mathbf{D_S}\right)^{1/2}$$

but, now, it is clear that all the eigenvalues appear in the complex-conjugate pares, allowing us to retain eigenvalues with either positive sign at the imaginary part, as in (90), or retain eigenvalues with negative sign. The same remark concerns eigenvectors in the Theorem 3.12.1.b.

Concluding remark 3.12.

The subsequent analysis of the system without damping can be done by applying methods developed in the preceding sections for the system with damping. These methods are based on either n, or $2n$-dimensional formalism.