MODELING OF DAMAGE IN ELASTIC PLASTIC BODIES

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Numerical methods for simulation of damage processes are discussed and implementation of very simple theory and numerical method is described. The results of numerical calculations are presented for damage process under extension of elastic-plastic specimen of rectangular section with circular and elliptic pores and rigid inclusions. The influence of physical and numerical parameters on results of calculations is investigated and definite recommendations on the numerical modeling of such processes are worked out.

1. Introduction

The process of destruction is treated as the phenomenon of accumulation of micro-defects such as micro-cracks and micro-pores that is finally causes formation of macro-cracks and fragmentation of solids (separation on the parts).

Consider basic approaches to the problems of destruction modeling. In the simplest approach the cracks are not examined, and the calculation of destruction is reduced to the estimation of the safety factors or to comparison of maximum design stresses and their limiting values predicted by the so-called material strength theories [1].

In many cases not only the evaluation of strength and safety factors is of interest, but also the prediction of the scenarios of the process of destruction, the determination of the time to macro-destruction, the study of the influence of various changes in the design and in material properties onto development of internal damages. This interest stimulated the development of the fracture mechanics.

In fracture mechanics it is assumed that the propagation of crack occurs under fulfilling of the criterion of destruction, formulated in terms of the coefficients of stress intensity. The calculations of the propagation of cracks are performed with the explicit account the newly formed crack surfaces with the use of methods of boundary or finite elements. The survey of publications on the realization of fracture mechanics approach and also the examples to realization are given in [2].

At present time preference is often given to the methods of through calculation and behavior of destroyed and not-destroyed computational cells (elements) are governed by unified algorithm. In this approach there is no need in explicit consideration of macro-cracks. The macro-cracks are modeled by the chain-lines of destroyed elements. Destruction development is realized by means of the stressed state correction in the destroyed elements. For the first time the numerical realization of this approach is made in [3] and then such approach received wide acceptance because of its simplicity and effectiveness. Now many modifications of this approach are known, such as the method of the collapse of the destroyed computational cells [4], method of the automated introduction of additional nodes with the formation of free surfaces in the destroyed elements [5]. The calculations of the stress-strain state in the not-destroyed elements are performed by using the usual equations of the theory of elasticity and plasticity. The integral effects of the material destruction, or effects of decrease of the material resistance to deformation in the calculations are the result of increase of the quantity of destroyed elements.

Experimental stress-strain history diagrams show the presence of the material resistance weakening intervals, which is characterized by the decrease of stresses under increase of deformations. However, such diagrams with the intervals of the material weakening cannot be considered as the characteristics of material for use in rheological (constitutive) equations for material media in calculations of the phenomena of the destruction. To this there are two reasons. The first reason consists in the fact that the material resistance weakening intervals characterize by no means the behavior of material in the infinitely small volume, they just demonstrate the integral diagram of the deformation of experimental specimen. In reality here we observe not rheological (material) but structural instability. The second reason consists in the fact that the initial-boundary value problems for the stress-strain diagrams with material weakening intervals become incorrect.

The equations of classical elastic-plasticity can be written as:

$$\partial_t \boldsymbol{\sigma} = E : (\partial_t \boldsymbol{\varepsilon} - \partial_t \boldsymbol{\varepsilon}_p)$$
$$\partial_t \boldsymbol{\varepsilon} = (\partial_x \mathbf{v})_s, \qquad \partial_t \boldsymbol{\varepsilon}_p = H(\|\boldsymbol{\sigma}\| - \sigma_s)\lambda(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) : \partial_t \boldsymbol{\varepsilon}$$
$$\rho \partial_t \mathbf{v} = \partial_x \boldsymbol{\sigma}$$

or

$$\begin{cases} \partial_t \boldsymbol{\sigma} = \mathbf{E}_{(ep)}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p) : \partial_x \mathbf{v} \\ \rho \partial_t \mathbf{v} = \partial_x \boldsymbol{\sigma} \end{cases}$$
$$\partial_t \boldsymbol{\varepsilon} = (\partial_x \mathbf{v})_s, \qquad \partial_t \boldsymbol{\varepsilon}_p = H(\|\boldsymbol{\sigma}\| - \sigma_s)\lambda(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) : \partial_t \boldsymbol{\varepsilon}$$

Here traditional tensor notation is used, symbol ":" designates the double inner product of tensors. Initial boundary value problems become incorrect because the Hadamard criterion

$$\partial_t \sigma : \partial_t \varepsilon = (\mathbf{E}_{(ep)} : \partial_x \mathbf{v}) : \partial_x \mathbf{v} > 0$$

is violated in the material resistance weakening intervals on stress-deformation history diagrams. The Hadamard inequality is also known in mechanics of solids as Drucker stability criterion.

The violation of the Hadamard criterion indicates the loss of the continuous dependence of the solution on the input information, and it means also the loss of the property of hyperbolicity of equations in the dynamic problems and the loss of property of ellipticity in the static problems. In the numerical calculation in this case the arithmetic overflow occurs or results become senseless demonstrating the absence of convergence.

In the mentioned above works [3-5] devoted to the calculation of destruction the instantaneous correction of stresses and the instant formation of the new surfaces of the growing crack are used. The interval of the material resistance weakening is replaced by the instant change of stress-strain state so it does not calculated and force no difficulties.

For processes of the ductile fracture with gradual material resistance weakening usual elastic-plasticity is not applicable. In order to make theory of elasticity and plasticity suitable for describing the material resistance weakening intervals, in [6, 7] it was proposed to associate the degradation of the properties of elasticity under destruction with the growth of special parameter of destruction that has been called damage.

There is a need in special additional independent parameter of state that is responsible for destruction because formation and accumulation of micro-defects occur not only with reaching of the critical stress-strained state, but also because of reasons or actions of non-thermomechanical nature (as example, laser radiation, chemical reactions and similar). Therefore the accumulation of damage (micro-pores and micro-cracks) is treated as the thermodynamically independent process. It is assumed that the damage of the intact material is equal to zero and grows proportionally to the accumulation of the micro-defects. The thermodynamic independence of the process of destruction from the processes of heating and deformation does not exclude their reciprocal influence.

Tensor nature of damage parameter can be various. It may be a scalar (most frequently) or (sometimes) the tensor of the second or fourth ranks [8].

With growth of damage parameter the elastic characteristics of material, such are the modules of elasticity and yield stress, tend to zero, that indicates the complete destruction of the structured material. With the damage accumulation the new internal macro-surfaces are formed and this leads to appearance and growth of macro-cracks that finally separate the solid body into the parts or even onto the separate material particles. The surveys of experiments and the detailed consideration of different variations of the theory of damaging may be found in the works [8-18] whose major results are examined below.

The introduction of the independent thermodynamic parameter of damage as the value responsible for the decrease of elastic constants removes the problem of the incorrectness of initial boundary value problems under the resistance weakening. Actually, the equations of the theory of damage take the form of the modified equations of the elastic-plasticity

$$\partial_t \boldsymbol{\sigma} = \mathbf{E}_{(ep)} : \partial_x \mathbf{v} + (\partial_\theta \mathbf{E} \partial_t \gamma) : (\mathbf{E}^{-1} : \boldsymbol{\sigma}) + \partial_x \cdot (\nu_\sigma \partial_x \boldsymbol{\sigma}),$$
$$\rho \partial_t \mathbf{v} = \partial_x \boldsymbol{\sigma} + \partial_x \cdot (\rho \nu_v \partial_x \mathbf{v}),$$
$$\partial_t \gamma = H(\Phi_\gamma) \lambda_\gamma (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \gamma) + \lambda_\gamma^{(0)}(x,t) + \partial_x \cdot (\nu_\gamma \partial_x \gamma),$$

where the symbol «$\cdot$» designates the inner product of tensors; γ is a damage parameter; $\Phi_\theta(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \gamma, ..., T, ...) \geq 0$ is a criterion for damage initiation; additional terms with second spatial derivatives are commented below. The third of the given equations describes the kinetics of damage. In this equation the first term corresponds to the increase of damage as a result of heating and deformation, the second term describes the increase of damage due to the non-thermo-mechanics actions.

The Drucker criterion in the theories of damage for weakening processes is violated (deformation process becomes unstable):

$$\partial_t \boldsymbol{\sigma} : \partial_t \boldsymbol{\varepsilon} = (\mathbf{E}_{(ep)} : \partial_x \mathbf{v}) : \partial_x \mathbf{v} + (\partial \boldsymbol{\sigma} / \partial \gamma) \partial_t \gamma : \partial_x \mathbf{v} \leq 0$$

Here first term with elastic-plastic operator $\mathbf{E}_{(ep)}$ is non-negative as in regular theory of elastic-plasticity, but second term with increase of damage is negative and even larger in the module than first term. So now Hadamard criterion is fulfilled

$$(\mathbf{E}_{(ep)} : \partial_x \mathbf{v}) : \partial_x \mathbf{v} > 0$$

and initial boundary value problem for velocities and stresses is correct.

In contrast to integral (experimental) diagrams with resistance weakening sections the local diagrams of deformation (or any other form of the law of hardening) for infinitesimal material volumes have no sections of the weakening and can be successfully used in calculations of elastic and plastic deformations.

The terms with the second spatial derivatives in the equations of theory of elastic-plasticity and damage are absent. They are used in gradient and integral theories [18] of elastic-plasticity of damaged materials. Multipliers ν with indexes σ, v, γ are the coefficients of gradient viscosity.

The additional terms in gradient theories introduce parabolic properties to the equations of the elastic-plastic theory of damage, improve properties of equations, smooth out the solutions and are, thus, regularizers (i.e., terms that improve the conditioning of equations). The physically substantiated magnitudes of the coefficients of gradient viscosity are so low that they cannot serve as a regularizer. Therefore in calculations the coefficients of such viscosity are artificially increased to the values, which do not violate the usual stability limitations for time step of explicit difference schemes.

In general case the role of the parameter, which is independent on deformation and is simultaneously responsible for the degradation of the elastic properties, may be delegated to some other state parameters, such as, for example, temperature or porosity without the risk of the loss of the correctness of initial boundary value problems. In models of elastic-visco-plastic materials increments of the plastic deformation are not proportional to increments of total deformation and therefore in such models the degradation of elastic properties may depend directly on the plastic deformation.

The initial boundary value problems using theory of elastic plastic flow with damage are solved by using the usual incremental method with the diagrams of deformation without the sections of the resistance weakening, but with the variable values of the elastic parameters in the dependence on the damage parameter.

The effect of the resistance weakening is not embedded into the mathematical model in the form of the dependence of yield stress from the deformation, but it is obtained in the extended space of the variables of state as the result of a growth in additional damage parameter and of degradation of the elastic properties of material. Thus, from a mathematical point of view introduction of damage parameter is the regularization of the initial-boundary-value problems of elastic-plasticity for calculating the unstable regimes of deformation.

In the calculations according to the theories of damage the development of the zones of destruction with the lowered elastic strength of materials is observed. In such zones the intensity of deformations and damage show splash, and displacements and speed undergo abrupt change (jump), that imitates the divergence of the coasts of macrocracks. Correctness of boundary-value problems in this case is supported, but their conditionality with the development of the zones of destruction deteriorates, which can lead (if we do not take preventive measures) to the pathologic dependence of the numerical solution on computational parameters such as grid, particle distribution and value of time step.

Of course the results of numerical simulation always depend on the parameters of discreteness, but with the mesh refinement or with the growth of the number of basis functions these solutions must demonstrate the convergence of the solutions to certain, not depending on the method, limits. The convergence and authenticity of numerical solutions must be established by the purely mathematical means, which include the a priori analysis of methods applied to the simplified tasks and the a posteriori testing of convergence to the known analytical solutions, which always can be obtained artificially. For this the arbitrary solution is substituted into the equations and the boundary conditions of task. Then the obtained discrepancies are used as known right sides of equations and conditions. For such new initial boundary value problem our arbitrary solution becomes exact and analytical. Solving obtained new initial boundary value problem the numerical method must reproduce chosen arbitrary solution with the adequate accuracy.

The strong dependence of the solution on the parameters of the discreteness such as size and the form of grid cells (or, in the more overall meaning, the dependence of the solution on the choice of basis functions) is the problem of the majority of numerical methods especially under calculation physically unstable regimes.

For the sake of fairness it is necessary to note that the absence of convergence in details or, more clearly to say, the absence of the repeatable picture of phenomenon in details is a characteristic feature of physical experiments on the destruction. This is explained by the sensitivity of the scenarios of the development of damages to the heterogeneity of the properties of real media and by the sensitivity to the small variations in the external actions.

The repeatable and reproducible characteristics of the processes of destruction can be may be represented by their integral characteristics, for example, integral (averaged on the calculated object) diagrams of deformation, the energy, spent on the destruction of the body being deformed, the time of its life. Results for such integral characteristics in the numerical modeling and in the experiments must demonstrate convergence. For this in the numerical simulation it is necessary to take all measures for regularization of the tasks under conditions of destruction, without distorting, as far as possible, the solution itself.

Above have been already noted the basic methods of the regularization of the boundary-value problems of the elastic-plasticity under conditions of resistance weakening.

To account the degradation of the material elasticity properties the first method uses parameter of damage that is independent on the deformation.

The second method is based on the theory of elasto-visco-plasticity and connects the weakening of the material elasticity properties under the destruction with increase in the viscoplastic deformation that also is independent on the deformation.

The third method of regularization is used in the gradient theories of the damaged elastic-plastic materials. An improvement of initial-boundary value problem conditionality is reached by the physically substantiated averaging of the dependent variables (stresses, deformations and of damage) in neighborhood of each point. This is equivalent to the smoothing of the unknowns by adding diffusive terms.

The fourth method is additional to each of three, mentioned above, and consists in accounting of the inertial forces in the tasks of destruction independently of the speed of loading. The accounting of inertia in the problems of elliptical type prevents the degeneration of boundary-value problems for the fragments, which were separated from the initial body under destruction. Actually, under destruction the body being deformed is divided by the non-interacting parts. Static boundary-value problems become incorrect for the parts deprived of fastenings. Therefore the inertia, that removes this drawback, should be provided for in the mathematical models of the calculation of destruction from the very beginning.

For guaranteeing the physical authenticity of the calculations of destruction important is the selection of the criterion of destruction and describing the kinetics of the damage parameter. This problem is examined into [17].

Important question is the choice of numerical method to use for destruction modeling. It is considered that the explicit method are effective in the tasks of wave propagation dynamics, and implicit methods are preferable in the tasks of quasi-statics. Recently, because of the appearance of the high performance computers, this opinion changes. The quasi-static problems of deformation and continuous destruction successfully are solved with the use of explicit diagrams [5] and, on the contrary, in the dynamic problems effectively are used implicit methods [19].

In spite of abundance published studies on the simulation of destruction, until now there are no clarity in the selection of mathematical model as well as in the construction of the methods of regularization and implementation. There are more questions, than answers. In the presented brief survey not all existing approaches have been considered. Here just illuminated the major existing directions in modeling of brittle and ductile fracture.

2. Formulation of the problem

The complete system of equations for the simulation of destruction, utilized in the present work, is the usual system of equations of the theory of elastic-plastic flow, augmented by kinetic equation for the damage and by dependence of the modules of elasticity and yield point from the damage. Temperature effects are not examined. For the small deformations this system of equations takes the form:

$$\rho\partial_t^2\mathbf{U} = \nabla\cdot\boldsymbol{\sigma}, \qquad \boldsymbol{\sigma} = \mathbf{E}(\gamma):(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_p), \qquad \boldsymbol{\varepsilon} = 1/2(\nabla\otimes\mathbf{U} + (\nabla\otimes\mathbf{U})^T)$$

$$\partial_t\boldsymbol{\varepsilon}_p = \lambda_p \frac{\partial F_p}{\partial\boldsymbol{\sigma}} H(F_p)H(\boldsymbol{\sigma}:\partial_t\boldsymbol{\varepsilon}), \qquad F_p(\boldsymbol{\sigma}) = 3/2(\boldsymbol{\sigma}':\boldsymbol{\sigma}')/\sigma_p^2 - 1$$

$$\partial_t\theta = H(F_d)\Gamma(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}_p,\gamma) + r_\gamma, \; F_d = F_d(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}_p,\gamma)$$

where ρ is a density; $\mathbf{U}$ is a displacement vector; $\boldsymbol{\sigma}$ is a stress tensor; $\boldsymbol{\sigma}' = \boldsymbol{\sigma} - (\boldsymbol{\sigma}:\mathbf{I})\mathbf{I}/3$ is a stress deviator; $\mathbf{E}(\gamma)$ is an elastic module tensor, which depends on scalar damage γ; $\boldsymbol{\varepsilon}$ is a strain tensor; $\boldsymbol{\varepsilon}_p$ is a plastic strain tensor; symbol «$\otimes$» designates external tensor product; λ_p - coefficient in flow rule, which is defined by plasticity condition $F_p(\boldsymbol{\sigma}) = 0$; F_p - plastic loading function; H - Heaviside function, which is equal to zero for the negative values of argument and to one otherwise; σ_p - yield limit; $\mathbf{I}$ - the unit tensor; F_d - damage condition function, non-negative values of which permit the accumulation of the damage; r_γ - the non-thermomechanical (for example, chemical) source of the damage. System of equations is supplied with main boundary conditions

$$\mathbf{U}\cdot\boldsymbol{\tau}_\alpha\big|_{\mathbf{x}\in\partial V_{u\alpha}} = U^*_\alpha(\mathbf{x},t),\ (\alpha=1,2)$$
$$\mathbf{U}\cdot\mathbf{n}\big|_{\mathbf{x}\in\partial V_{un}} = U^*_n(\mathbf{x},t)$$

with natural boundary conditions

$$\boldsymbol{\sigma}:\mathbf{n}\otimes\boldsymbol{\tau}_\alpha\big|_{\mathbf{x}\in\partial V_{\tau\alpha}=\partial V\setminus\partial V_{u\alpha}} = p^*_\alpha(\mathbf{x},t)$$
$$\sigma:\mathbf{n}\otimes\mathbf{n}\big|_{\mathbf{x}\in\partial V_n=\partial V\setminus\partial V_{un}} = -p^*_n(\mathbf{x},t)$$

and with initial conditions:

$$\mathbf{U}\big|_{t=0} = \partial_t\,\mathbf{U}\big|_{t=0} = \boldsymbol{\varepsilon}_p\big|_{t=0} = \gamma\big|_{t=0} = 0$$

where $\mathbf{n}$ and $\boldsymbol{\tau}_\alpha$ ($\alpha=1,2$) are the unit normal and tangent vectors to the boundary. The predefined functions are marked by asterisks.

3. Method of the solution

Tasks are examined in the two-dimensional setting under the plane strain conditions using variant of final element method [19]. For all unknown functions is used piecewise-linear approximation on the triangular and quadrangular elements. For step-by-step solution in time the two layer implicit Euler scheme is used. Time step magnitude is limited to accuracy condition in order the solution increment at each step in time be much less than the solution itself.

At each time step the usual linearized boundary-value problem of the theory of elastic-plasticity relative to displacement increments is solved:

$$\rho\frac{2}{\Delta t^n}\left(\frac{\Delta\mathbf{U}^{n+1}}{\Delta t^n}-\mathbf{v}^n\right) = \nabla\cdot(\boldsymbol{\sigma}^n+\Delta\boldsymbol{\sigma}(\Delta\mathbf{U}^{n+1})) \qquad (\mathbf{x}\in V)$$
$$(\mathbf{U}^n+\Delta\mathbf{U}^{n+1})\cdot\boldsymbol{\tau}_\alpha\big|_{\mathbf{x}\in\partial V_{u\alpha}} = U^*_\alpha(\mathbf{x},t^{n+1}) \qquad (\alpha=1,2)$$
$$(\mathbf{U}^n+\Delta\mathbf{U}^{n+1})\cdot\mathbf{n}^n\big|_{\mathbf{x}\in\partial V_{un}} = U^*_n(\mathbf{x},t^{n+1})$$
$$(\sigma^n+\Delta\sigma(\Delta\mathbf{U}^{n+1})):\mathbf{n}^n\otimes\boldsymbol{\tau}^n_\alpha\big|_{\mathbf{x}\in\partial V_{\tau\alpha}=\partial V\setminus\partial V_{u\alpha}} = p^*_\alpha(\mathbf{x},t^{n+1}) \qquad (\alpha=1,2)$$
$$(\sigma^n+\Delta\sigma(\Delta\mathbf{U}^{n+1})):\mathbf{n}^n\otimes\mathbf{n}^n\big|_{\mathbf{x}\in\partial V_n=\partial V\setminus\partial V_{un}} = -p^*_n(\mathbf{x},t^{n+1})$$

where $\Delta\boldsymbol{\sigma}(\Delta\mathbf{U}^{n+1})$ is a linear differential operator that converts displacement increments into incremental stresses in accordance with elastic-plastic flow theory. This boundary value problem after discretization is solved using the matrix-free conjugate gradient method [19]. Then displacement increments are used to calculate deformations, plastic deformations and stresses. Finally the kinetic equation for the damage parameter is integrated.

Some special features of destruction modeling should be noted here. Before the destruction starts the deformation process may be slow (quasi-static deformation)/ It means that the inertia terms in equations of motion may be neglected. However, with the appearance of the zones of destruction the process of deformation is capable sharply to be accelerated, becoming dynamic. In advance to predict the time and place onset of zones of destruction without conducting of calculation is impossible; therefore the inertia terms in the algorithm of the solution is necessary from the very beginning. Without inertia the correctness of boundary-value problems for destroyed and separated parts is violated and calculations of their evolution will be impossible.

Used numerical methods are approximate and therefore the calculated sought functions may have nonphysical small-scale oscillations in the space and time with the lengths of half-waves, equal to the steps of time-spatial grid. Such non-monotonicities are revealed by the change of the sign of the second derivatives of the function along definite coordinate at the ends of grid rib. The directions of the rib and coordinate may be differ from each other. Such nonphysical non-monotonicities must immediately be corrected, otherwise under the conditions of the poor conditionality of the tasks they can distort numerical solution, causing the appearance of false zones of destruction.

The monotonization of the sought functions is conducted into two stages. During the first stage is used “physical” method of smoothing, based on the introduction into the evolutionary equations for the plastic deformations and the damage parameter of the small viscous terms:

$$\partial_t \boldsymbol{\varepsilon}_p = \lambda_p \frac{\partial F_p}{\partial \boldsymbol{\sigma}} H(F_p) H(\boldsymbol{\sigma} : \partial_t \boldsymbol{\varepsilon}) + \nabla \cdot (\nu_p \nabla \boldsymbol{\varepsilon}_p)$$
$$\partial_t \gamma = H(F_d) \Gamma(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p, \gamma) + r_\gamma + \nabla \cdot (\nu_\gamma \nabla \gamma)$$

where ν_p, ν_γ are the coefficients of viscosity. To a question about the magnitude of the viscosity coefficients single-valued answer is not given by neither theory nor experiment. It is possible to expect that the experimental physical values of the coefficient of viscosity will be too small for guaranteeing the effective monotonization of the solution with the actually utilized rough discretization. One way or another, depending on the explicit or implicit approximation of diffusion terms in equations, using the spatial one-dimensional model problems it is not difficult to find minimum values of the coefficient of the viscosities, necessary for decreasing the oscillations of numerical solutions. In our calculations this value was taken as: $\nu = c^2 \Delta t / 2$.

It is characteristic that introduced viscosity decreases the oscillations, but does not guarantee their absence. Therefore the solution is corrected additionally. In the second stage is used “mathematical” method of smoothing, which is consisted in the elimination of newly being appeared non-monotonicities by nonlinear smoothing. For this in the end of each step on the time for each sought function f the second derivative f_{xx} is calculated for each coordinate direction x from the solution of the following auxiliary problem:

$$\int_V (\nabla f \cdot \nabla \delta f_{xx} + f_{xx} \delta f_{xx}) dV = 0$$

$$f_{xx}\big|_{\partial V} = 0$$

i.e., the second derivatives thus are determined under the conditions of the simplest approximation of the solution by piecewise-linear functions. The matrix of system of equations for the second derivatives is diagonal due to the use of quadrature formulas with the points of integration in the mesh nodes.

If on a certain rib of grid the value f_{xx} the sign reverses, then in the adjacent nodes, which determine this edge, is carried out the local smoothing of the solution by the shift of the value of the function f_i to its average value in the direction x:

$$f_i := (f_i + (f(x_i - h) + f(x_i + h))/2)/2$$

where h is a small increment of coordinate x; values $f(x_i - h)$ и $f(x_i + h)$ are determined by interpolation. The second method, in contrast to the first, in the zones of the monotonous solution does nothing. Without explicit physical reasons in favor of the first method, the first stage may be fully eliminated from the procedure of smoothing.

The elastic properties of material are described by the simplified form of Hooke's law, obtained under the assumption of the initial isotropy of the material:

$$\sigma = \lambda \mathbf{I}(\varepsilon - \varepsilon_p) : \mathbf{I} + 2\mu(\varepsilon - \varepsilon_p)$$

where λ, μ are Lame elastic constants, σ_p is the yield stress that determines the boundaries of the elastic behavior of material. These elastic coefficients depend on damage as follows:

$\lambda = \lambda_0 e^{-1000\gamma}$, $\mu = \mu_0 e^{-1000\gamma}$, $\sigma_p = \sigma_{p0} e^{-1000\gamma}$.

Index «0» marks values for the undamaged material. By the local criterion the damage parameters is growing if the maximal principal deformation exceeds the positive critical value ε_d:

$$F_d = \frac{1}{2}\left[(\varepsilon_x + \varepsilon_y) + \left((\varepsilon_x - \varepsilon_y)^2 + 4\varepsilon_{xy}^2\right)^{1/2}\right] M - \varepsilon_d \geq 0$$

where M is the dimensionless scale factor

$$M = \sqrt{\frac{\min(h_x, h_y)}{\max[(x_{\max} - x_{\min}), (y_{\max} - y_{\min})]}}$$

This factor accounts the root rule of the concentration of deformations near the tip of crack in the elastic material. This factor provides imitation the using of coefficients of concentration of deformations. The numerical experiments show that this factor helps to decrease the dependence of damage criterion on the grid cells size.

The criterion of destruction with the scale factor ensures the convergence of the numerical values of the integral critical loads of destruction under the mesh refinement. Otherwise it turns out that the smaller the grid cells, the earlier begins the destruction, since the concentration of deformations is described much better on the grid of smaller cells and the critical levels of deformation are reached at the smaller values of the applied load. Certainly critical loads depend on grid permission, but this dependence must demonstrate the convergence of calculated critical loads to a certain limiting value, that also ensures scale factor. The introduction of coefficient is not the final solution of the problem of the convergence of the calculated critical loads of destruction, but it is possible that this is step to the side of the possible solution of this problem. It can be, someone will devise the best regularization of the criterion of destruction.

An important feature of the accepted criterion of destruction is that it distinguishes between extension and compression, reacts to a shear, since it is formulated in the main axes. So the criterion is not as trivial as it may seem. This criterion has been used to calculate origination and propagation of cracks in samples under shear and compression in [10]. As follows from the calculations, it works better than the criteria that are based on a limit values of invariants of strain tensors or stresses. Another feature of the criterion is that thanks to factor M it is formulated using approximation of the deformation concentration coefficients. This is the best choice, since total deformation, unlike stresses, elastic and plastic deformations, is not corrected when integrating the evolutionary constitutive relationships, but is uniquely determined by the kinematics of the deformation process.

In the absence of specific information on the kinetics of damage, the kinetic equation for the damaging parameter is adopted in the simplest form:

$$\partial_t \gamma = 1000 H(F_d)$$

where H is Heaviside function. The large coefficient on the right-hand side is introduced in order to provide a fast, but finite rate of growth of damage and consequently fast degradation of an ability of resistance to deformation during destruction (in a few time steps). This allows us to consider the mathematical model of fracture used here as a regularized and, at the same time, simplified version of the well-known model of the destruction [3], where at destruction the stress-strain state is corrected instantly.

4. Results

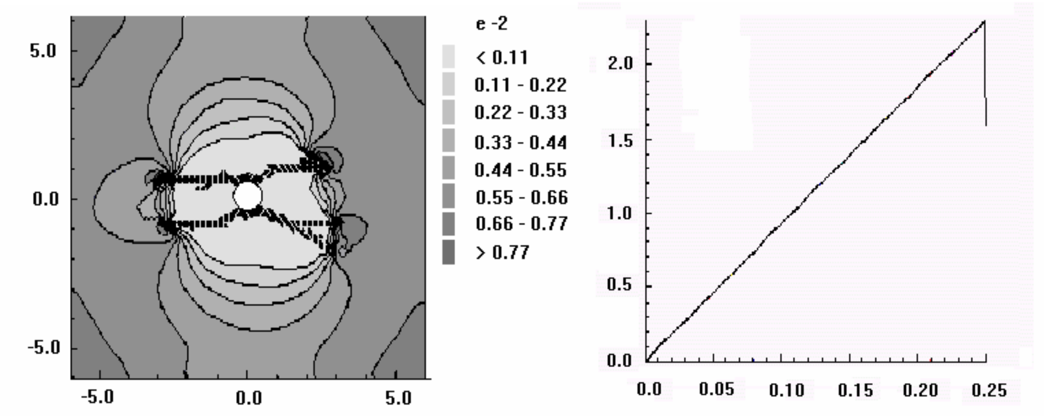
Consider the development of damage in specimens of rectangular section with pores or rigid inclusions of circular and elliptic shape under tensile loading.

The two-dimensional solution domain is the rectangle on the plane (x, y), having the circular or elliptic holes, which are treated either as macro-pores or as rigid inclusions. The grid of triangular finite elements is a usual almost uniform grid, generated automatically.

At the left and right boundaries the rectangle horizontal displacements are equal to zero. Lower boundary has zero vertical displacement, upper boundary slowly moves up with a speed much less than speed of sound in the material. This provides the quasi-static tension of the body in question. Boundary shear stresses are zero. Boundaries of macro-pores are free. The rigid inclusions had grid cells with "infinite" magnitudes of elastic constants and "infinite" yield stress. That imitates rigid material. Displacements at boundaries between rigid inclusions and elastic-plastic body were continuous.

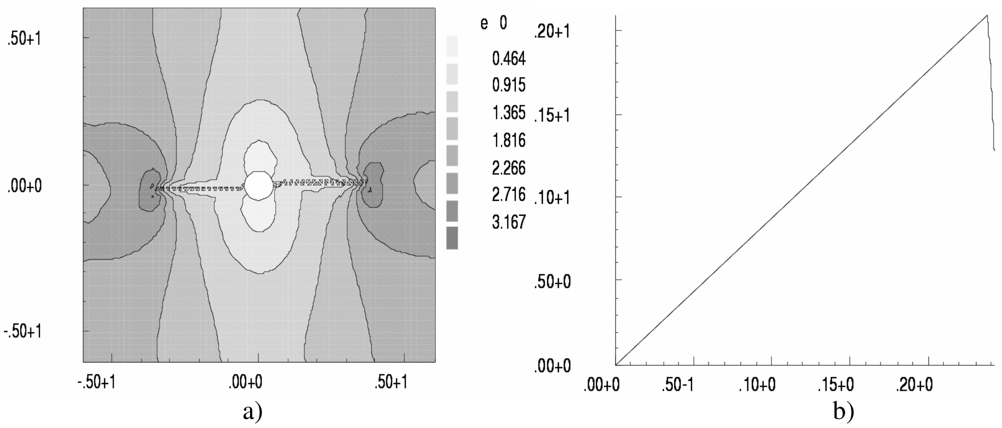
In all cases calculated diagrams of deformation depict the dependence of the stress averaged over the upper horizontal section on the monotonically increasing averaged total deformation (ratio of vertical displacement of upper boundary to the height of sample).

In dimensionless form the solution domain is defined by inequalities $-6.0 \le x \le 6.0$ and $-6.0 \le y \le 6.0$, the diameter of circular pore and circular rigid inclusion is equal to 1.0. The ratio of the semi-axes of elliptic pores is equal to 0.5, angle of rotation is equal to $30°$. Young's modulus is equal to 1000, Poisson ratio is equal to 0.2, the sound velocity is equal to 1.0, the velocity of upper boundary v_y grows with constant acceleration from zero at t=0 to the value 0.001 at $t = 10^5$. So the deformation process is quasi-static. In the calculations according to the model of ideally-plastic material the yield stress is equal to 1. The deformation of destruction is equal to 0.02.



a) b)

Figure 1. Circular rigid inclusion in the elastic material:
a) zones of damage and maximal principal deformation; b) average stress-strain history



a) b)

Figure 2. Damage of elastic material near the circular pore:
a) zones of damage and distribution of stress σ_y; b) average stress-strain history

The results for the elastic body with the circular rigid inclusion are shown in figure 1. Black narrow zones correspond to the developing macro-fissures, darkening answers the amount of maximal principal deformation. Right graph depicts the calculated stress-strain diagram. The case with one circular is demonstrated by time in figure 2.

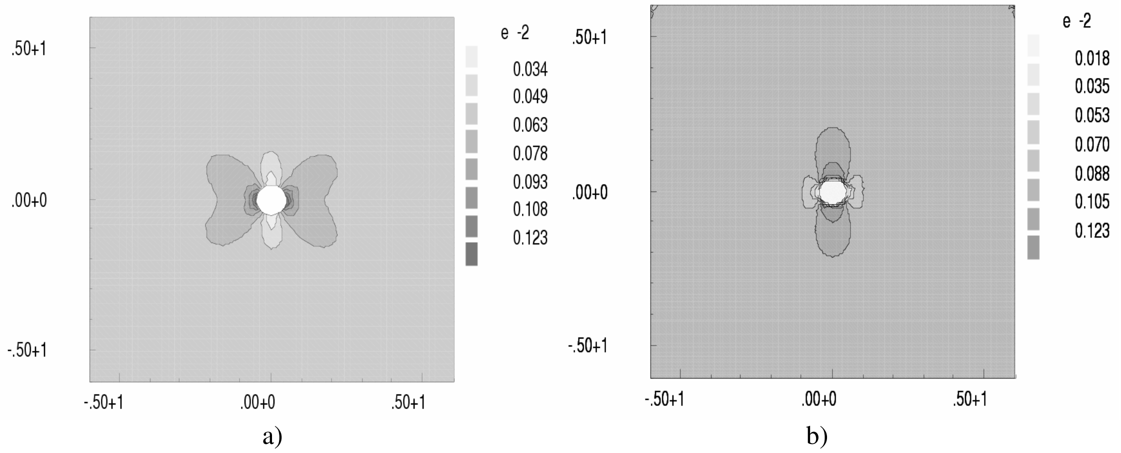


Figure 3. Maximal principal deformations in the elastic material
a) the pore and b) the rigid inclusion

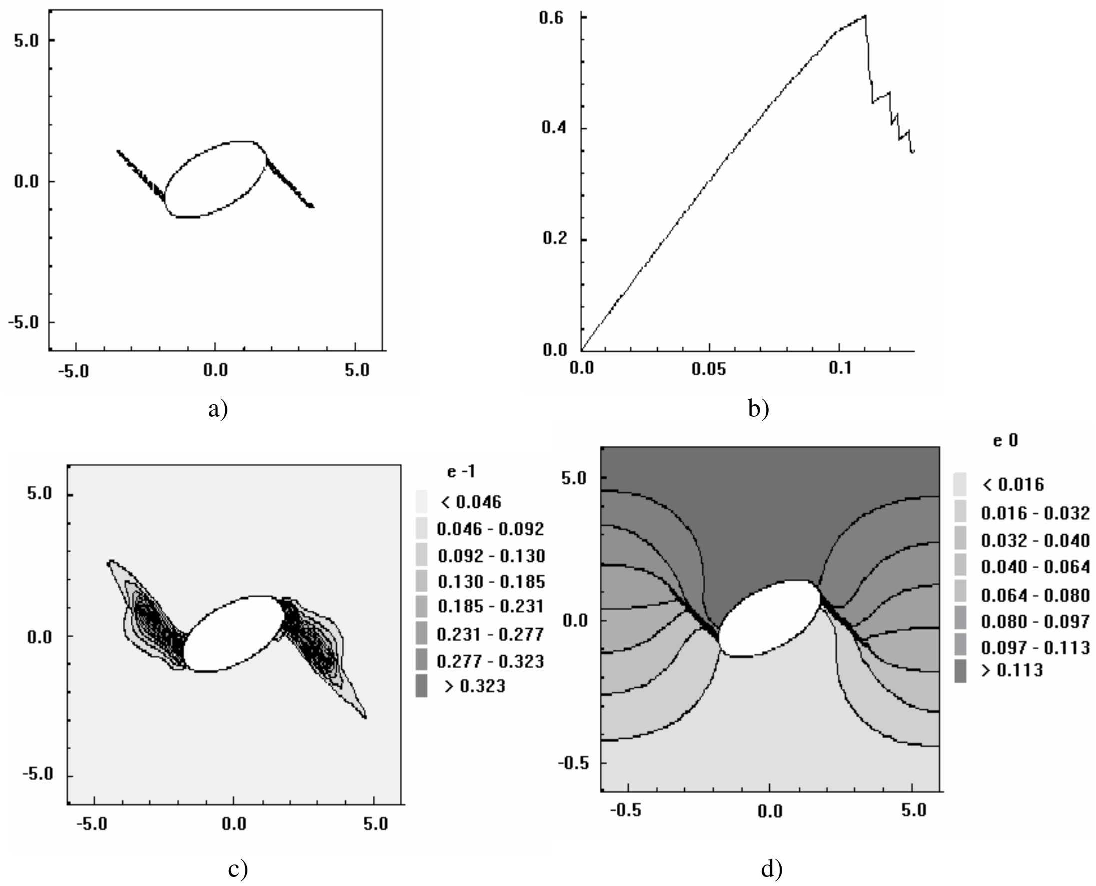
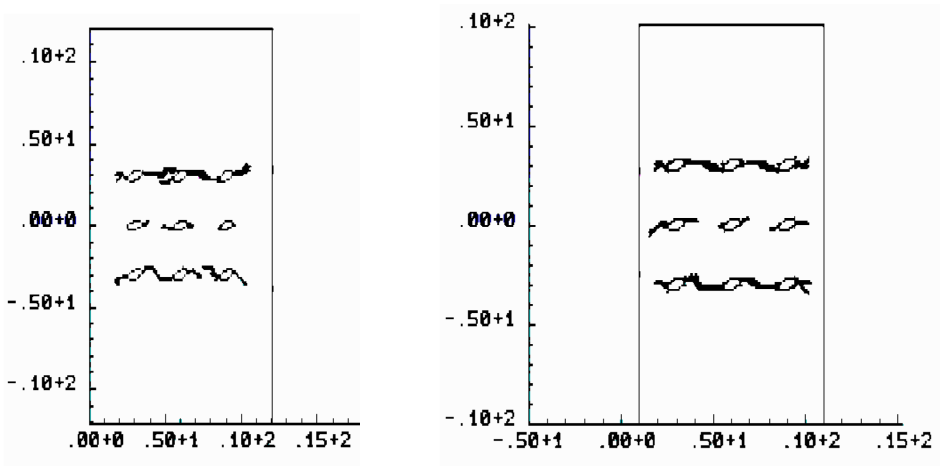


Figure 4. Destruction of ideal elastic-plastic material with one large elliptic pore:
a) zones of damage; b) average strain-stress history; c) plastic work; d) vertical displacement

A difference in the pictures of destruction is explained by the different nature of the concentration of deformations near the time and near the rigid inclusion (figure 3). In the case of the pore the destruction begins from the points in the horizontal direction x outermost from the center of the pore, and for the rigid inclusion destruction earlier begins at the points in the vertical direction y outermost from the center of inclusion.

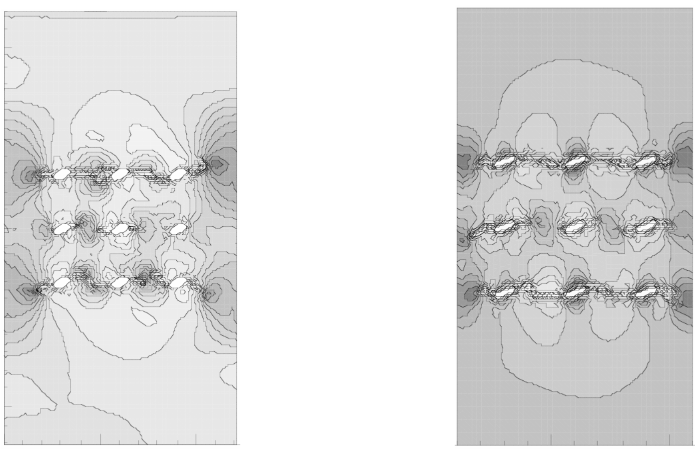
Solution for the elliptical pore, oriented at the angle $30°$, in the elastic-plastic material it is shown in figure 4, where are depicted the zone of destruction (a), the calculated diagram of deformation (b), the distribution of plastic work and vertical displacement. The splash of the values of plastic work in the zone of destruction and an abrupt change of the displacement is distinctly visible.

In figures 5-7 the results for damage of ideal elastic-plastic specimens with multiple elliptical pores and rigid inclusions, turned to the angle $30°$ are presented. The zone of damage can be seen in figure 5. The distribution of mean stress for the specimen with the pores and with the rigid inclusions can be seen in figure 6. Calculated stress-strain diagrams are depicted in figure 7. Figure 6 shows the unloading zones near the coasts of “cracks” and stress concentration near the tips of cracks.



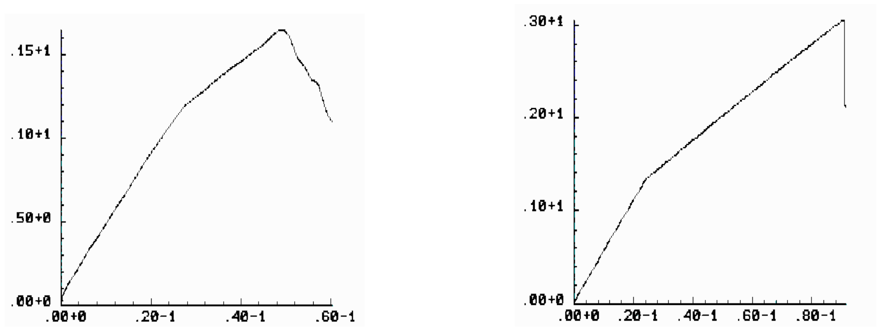
a) b)

Figure 5. Zones of damaged material for specimen: a) with the pores and b) the rigid inclusions



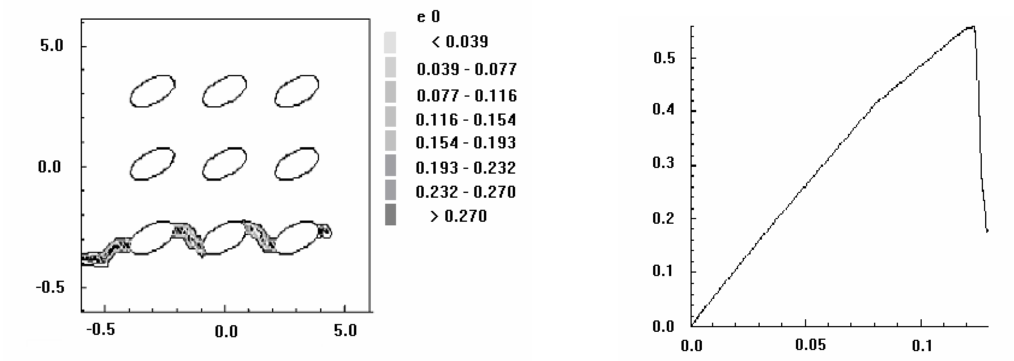
a) b)

Figure 6. The distribution of mean stress for specimen a) with pores and b) with rigid inclusions
(dark color corresponds to higher level of stresses)



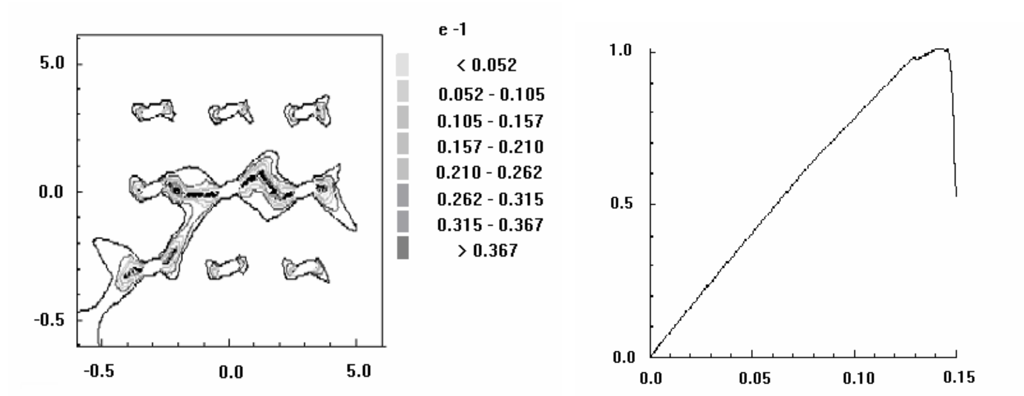
a *b*

Figure 7. History of stress-strain state in average for specimen: a) with pores; b) with rigid inclusions



a) b)

Figure 8. Deformation of elastic-ideal-plastic specimen with the group of turned elliptic pores in the absence the horizontal displacements of the lateral boundaries: a) zones of damaged material; b) calculated average strain-stress state diagram



a *b*

Figure 9. Damage of ideal-elastic-plastic material with the group of the small oval pores, oriented at the angle 30°, in the absence the horizontal displacements of the lateral boundaries: a) the zone of damage; b) the calculated average strain-stress diagram

The zones of damage and graphs of strain-stress state have been compared for cases of ideal-elastic-plastic material with big and small elliptical pores. The result is obvious: the sample with small pores is more resistant.

In parametric calculations investigated the effect of the characteristics of the sampling. As it turned out, depending on the size of the steps in space and time and depending on the shape of the mesh cells local picture of destruction varies in detail, while the history of strain-stress state of the sample in average is changed insignificantly

5. Conclusions

Numerical experiments with the simplest theory of damage show that:

- the convergence of numerical solutions is demonstrated for integral characteristics, such as diagram “averaged stress - averaged deformation”, while the local details of damage process strongly depend on mesh parameters;
- local damage criterion should be formulated in terms of strain intensity factors taking into account scale factor in order to ensure the convergence of limit loads;
- for the localization of strains in the form of narrow bands of contact discontinuities (“macrocracks”) the rapid loss of elastic resistance with the accumulation of damage is required, otherwise the zones of damage cover the substantial part of the solution region and strain localization is absent or very weak;
- it is necessary to support the monotony of the solution for eliminating the short wave oscillations of numerical solutions, otherwise such nonphysical oscillations can generate the false zones of damage;
- the retention of destroyed material resistance to volumetric compression is necessary in order to avoid degenerated grid cells appearance in the zones of damage;
- the accounting of the inertia forces regardless of the loading rate is necessary for retaining the correctness of initial boundary value problems during fragmentation;
- the control of accuracy by limiting the time step value is necessary for guaranteeing the convergence of the solutions “in the large” even with the use of unconditionally stable implicit schemes.

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